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A Reference to Perfect Numbers in Plato's Theaetetus

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For C.M.T., the Master

1. The Text

There are several mathematical passages in Plato's *Theaetetus*. The fictional description of Theodorus' lesson and the subsequent exposition of Theaetetus' findings about "powers" (147C–148B) have been analysed by many scholars,[1] while the following text appears to have attracted very little or no attention (*Theaetetus* 204A7–205A10):[2]

ΣΩ. Ὅτι οὗ ἂν ᾖ μέρη, τὸ ὅλον ἀνάγκη τὰ πάντα μέρη εἶναι. ἢ καὶ τὸ ὅλον ἐκ τῶν μερῶν λέγεις γεγονὸς ἕν τι εἶδος ἕτερον τῶν πάντων μερῶν;

ΘΕΑΙ. Ἔγωγε.

ΣΩ. Τὸ δὲ δὴ πᾶν καὶ τὸ ὅλον πότερον ταὐτὸν καλεῖς ἢ
204B ἕτερον ἑκάτερον;

ΘΕΑΙ. Ἔχω μὲν οὐδὲν σαφές, ὅτι δὲ κελεύεις προθύμως ἀποκρίνασθαι, παρακινδυνεύων λέγω ὅτι ἕτερον.

ΣΩ. Ἡ μὲν προθυμία, ὦ Θεαίτητε, ὀρθή· εἰ δὲ καὶ ἡ ἀπόκρισις, σκεπτέον.

ΘΕΑΙ. Δεῖ γε δή.

ΣΩ. Οὐκοῦν διαφέροι ἂν τὸ ὅλον τοῦ παντός, ὡς ὁ νῦν λόγος;

ΘΕΑΙ. Ναί.

ΣΩ. Τί δὲ δή; τὰ πάντα καὶ τὸ πᾶν ἔσθ' ὅτι διαφέρει; οἷον ἐπειδὰν λέγωμεν ἕν, δύο, τρία, τέτταρα, πέντε, ἕξ, καὶ

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[1] Recent scholarly contributions include e.g. Knorr 1975, especially chapters III to VI, Burnyeat 1978, Høyrup 1990a, Vitrac (forthcoming). The latter seems to me to settle the *vexata quaestio* of the meaning of δύναμις in 147C–148B.

[2] W. F. Hicken's text in the new Oxford edition is identical with Burnet's.

C ἐὰν δὶς τρία ἢ τρὶς δύο ἢ τέτταρά τε καὶ δύο ἢ τρία καὶ δύο καὶ ἕν, πότερον ἐν πᾶσι τούτοις τὸ αὐτὸ ἢ ἕτερον λέγομεν;

ΘΕΑΙ. Τὸ αὐτό.

ΣΩ. Ἆρ' ἄλλο τι ἢ ἕξ;

ΘΕΑΙ. Οὐδέν.

ΣΩ. Οὐκοῦν ἐφ' ἑκάστης λέξεως πάντα ἓξ εἰρήκαμεν;

ΘΕΑΙ. Ναί.

ΣΩ. Πᾶν δ' οὐδὲν λέγομεν τὰ πάντα λέγοντες;

ΘΕΑΙ. Ἀνάγκη.

ΣΩ. Ἦ ἄλλο τι ἢ τὰ ἕξ;

ΘΕΑΙ. Οὐδέν.

D ΣΩ. Ταὐτὸν ἄρα ἔν γε τοῖς ὅσα ἐξ ἀριθμοῦ ἐστι[3] τό τε πᾶν προσαγορεύομεν καὶ τὰ ἅπαντα;

ΘΕΑΙ. Φαίνεται.

ΣΩ. Ὧδε δὴ περὶ αὐτῶν λέγωμεν. ὁ τοῦ πλέθρου ἀριθμὸς καὶ τὸ πλέθρον ταὐτόν· ἦ γάρ;

ΘΕΑΙ. Ναί.

ΣΩ. Καὶ ὁ τοῦ σταδίου δὴ ὡσαύτως.

ΘΕΑΙ. Ναί.

ΣΩ. Καὶ μὴν καὶ ὁ τοῦ στρατοπέδου γε καὶ τὸ στρατόπεδον, καὶ πάντα τὰ τοιαῦτα ὁμοίως; ὁ γάρ ἀριθμὸς πᾶς τὸ ὂν πᾶν ἕκαστον αὐτῶν ἐστιν.

ΘΕΑΙ. Ναί.

E ΣΩ. Ὁ δὲ ἑκάστων ἀριθμὸς μῶν ἄλλο τι ἢ μέρη ἐστίν;

ΘΕΑΙ. Οὐδέν.

ΣΩ. Ὅσα ἄρα ἔχει μέρη, ἐκ μερῶν ἂν εἴη;

ΘΕΑΙ. Φαίνεται.

ΣΩ. Τὰ δέ γε πάντα μέρη τὸ πᾶν εἶναι ὡμολόγηται, εἴπερ καὶ ὁ πᾶς ἀριθμὸς τὸ πᾶν ἔσται.

ΘΕΑΙ. Οὕτως.

ΣΩ. Τὸ ὅλον ἄρ' οὐκ ἔστιν ἐκ μερῶν. πᾶν γὰρ ἂν εἴη τὰ πάντα ὂν μέρη.

ΘΕΑΙ. Οὐκ ἔοικεν.

ΣΩ. Μέρος δ' ἔσθ' ὅτου ἄλλου ἐστὶν ὅπερ ἐστὶν ἢ τοῦ ὅλου;

ΘΕΑΙ. Τοῦ παντός γε.

205A ΣΩ. Ἀνδρικῶς γε, ὦ Θεαίτητε, μάχῃ. τὸ πᾶν δὲ οὐχ ὅταν μηδὲν ἀπῇ, αὐτὸ τοῦτο πᾶν ἐστιν;

ΘΕΑΙ. Ἀνάγκη.

ΣΩ. Ὅλον δὲ οὐ ταὐτὸν τοῦτο ἔσται, οὗ ἂν μηδαμῇ μηδὲν ἀποστατῇ, οὗ δ' ἂν ἀποστατῇ, οὔτε ὅλον οὔτε πᾶν, ἅμα γενόμενον ἐκ τοῦ αὐτοῦ τὸ αὐτό;

ΘΕΑΙ. Δοκεῖ μοι νῦν οὐδὲν διαφέρειν πᾶν τε καὶ ὅλον.

ΣΩ. Οὐκοῦν ἐλέγομεν ὅτι οὗ ἂν μέρη ᾖ, τὸ ὅλον τε καὶ πᾶν τὰ πάντα μέρη ἔσται;

ΘΕΑΙ. Πάνυ γε.

The quotation is longer than strictly necessary, the arithmetical example concerning us being located at 204B–C. We shall see, however, that a careful analysis of the context will be decisive. The following translation is Burnyeat–Levett's, reworked at some points.[4]

SOC. Because when a thing has parts, the whole is necessarily all the parts. Or do you mean by 'the whole' also a single form arising out of the parts, yet different from all the parts?
THEAET. I do.
SOC. Now do you call 'all of it' and 'the whole' the same thing or different things?
THEAET. I don't feel at all certain; but as you keep telling me to answer up with a good will, I will take a risk and say they are different.
SOC. Your good will, Theaetetus, is all that it should be. Now we must see if your answer is too.
THEAET. We must, of course.
SOC. As the argument stands at present, the whole will be different from all of it?
THEAET. Yes.
SOC. Well now, is there any difference between all of them and all of it? For instance, when we say 'one, two, three, four, five, six'; and 'twice three' or 'three times two' or

[3] A similar expression can be found in *Cratylus* 432A8.
[4] Burnyeat 1990, pp. 342–343. Burnyeat–Levett sometimes employ fairly different renderings even if the Greek text carries almost identical expressions. In those instances I have kept the translation uniform. Most importantly, τὸ πᾶν and τὰ πάντα have been translated as "all of it" and "all of them". (The suggestion is by N. Denyer and is adopted in Harte 2002, cfr. note 75 on p. 40.) Burnyeat–Levett's translation "the sum" of the first noun–phrase carries in fact unjustified arithmetical overtones.

'four and two' or 'three and two and one', are we speaking of the same thing in all these cases or different things?

THEAET. The same thing.

SOC. Anything other than six?

THEAET. Nothing.

SOC. Then with each expression have we not spoken of all the six?

THEAET. Yes.

SOC. And when we speak of all of them, aren't we speaking of all of it?

THEAET. We must be.

SOC. Anything other than the sextet?

THEAET. Nothing.

SOC. Then in all things made up of number, at any rate, by 'all of it' and 'all of them' we mean the same thing?

THEAET. So it seems.

SOC. Now let us talk about them in this way. The number of an acre is the same thing as an acre, isn't it?

THEAET. Yes.

SOC. Similarly with a mile.

THEAET. Yes.

SOC. And the number of an army is the same as the army; and so always with things of this sort? Their total number is in fact all of what each of them is.

THEAET. Yes.

SOC. But is the number of each anything other than its parts?

THEAET. No.

SOC. Now things which have parts consist of parts?

THEAET. So it seems.

SOC. And it is agreed that all the parts are all of it, since the total number is to be all of it.

THEAET. That is so.

SOC. Then the whole does not consist of parts. For if it did, it would be all the parts and so would be all of it.

THEAET. It looks as if it doesn't.

SOC. But can a part, as such, be a part of anything other than the whole?

THEAET. Yes, of all of it.

SOC. You are putting up a good fight anyway, Theaetetus. But all of it now – isn't it just when there is nothing lacking that it is all of it?
THEAET. Necessarily.
SOC. And won't this very same thing – that from which nothing anywhere is lacking – be a whole? While a thing from which something is absent is neither a whole nor all of it – the same consequence having followed from the same condition in both cases at once?
THEAET. Well, it doesn't seem to me now that there can be any difference between all of it and whole.
SOC. Very well. Now were we not saying that in the case of a thing that has parts, both the whole and all of it will be all the parts?
THEAET. Yes, certainly.

The text at 204B–C could be taken as presenting a trivial piece of mathematics, whose sole function is to clarify a subtle philosophical point (whether a whole is the same as its parts or not) by a very elementary example. I shall argue instead that such a reading of the passage is simplistic: my proposal is that the text contains the first known reference to perfect numbers. Surprisingly enough, such a proposal appears to be made for the first time in the present paper. In Section 2 a number of arguments supporting this thesis are presented; in particular the context is analysed in which the passage is inserted. In Section 3 the value of this new testimony for our knowledge of pre–Euclidean mathematics is briefly assessed.

N.**B**. "Perfect numbers" is a very infelicitous translation, given the emphasis on value the denomination carries. An unpretentious "complete numbers" would be far better, as C. M. Taisbak suggested years ago.[5] In the present paper I will use the denomination "complete numbers" when referring to the notion in the context of the ancient mathematical *corpus*.

2. The context

[5] See Taisbak 1976, p. 273.

It is plain that the passage carries some mathematical content. The problem is to determine the extent and the relevance of such a content. The following possibilities could be envisaged:

i. The example at 204B–C is random: its mathematical content does not go beyond what is explicitly stated.

ii. The example is purposedly chosen; its mathematical content is significant.

A problem with such a dichotomy is that it is not obvious what counts for an example to have a significant mathematical content. Let us take, for the sake of comparison, the use of 5+7=12 as a paradigmatic addition (and of 5+7=11 as a paradigmatic miscalculation) at *Theaetetus* 196A and 199B. The same example 5+7=12 can be found in Kant's *Kritik der reinen Vernunft* as an instance of arithmetical statement,[6] i.e. of what Kant regards as a typical *a priori* synthetic judgement, a central notion in his system. It is very likely that Kant's choice had been directed by the primary choice made by Plato, suggesting that reference to tradition is an essential feature of philosophy. In this respect, it is at all significant that the example is drawn from the Platonic dialogue dealing expressly with knowledge. In Kant's instance there is thus a serious, albeit unexpressed, philosophical motivation under the mathematically trivial surface. The presence of an opaque motivation, from the philosophical if not from a mathematical point of view, could be assumed also in Plato's original example 5+7=12. But this is an empty conjecture unless some external or contextual clue is found for determining what motivation there was, and this does not happen to be the case in this instance.[7] A mathematical example in a non–mathematical writing has seldom a significant content in itself. Such a content can be uncovered only when the example is related to the context in which it is inserted or to a whole tradition of thought.

[6] I. Kant, *Kritik der reinen Vernunft*, Introduction, Section V.

[7] In philosophical writings, the sum 5+7=12 became a stock example of a simple arithmetical equality: it is to be found in § 2 of Frege's *Die Grundlagen der Arithmetik*, but this is its only occurrence in the treatise. 5+7=12 (possibly miscalculated as 5+7=11 as in the *Theaetetus*) is the only example in Dummett's paper Wittgenstein's Philosophy of Mathematics (Dummett 1959). One could therefore guess that 5+7=11 is to be found as an example of wrong calculation in some of Wittgenstein's notes on philosophy of mathematics, e.g. in his 1939 *Lectures on the Foundations of Mathematics.* Those lectures could really have been a good place, since the problem of making a mistake in computing is extensively discussed, and since Wittgenstein devoted particular attention to the *Theaetetus*, most notably to the theory expounded in the dream and to the immediately subsequent criticisms of it (see Wittgenstein 1953, § 46 ff.). But the example is not there (but see *Lecture* 8). But even if the example could be found in the 1939 *Lectures*, we should not be entitled to ascribe any intention of oblique philosophical reference to the author: the fact is simply that in the 1939 *Lectures* there are plenty of examples of wrong numerical calculations, whereas e.g. in Kant's *Kritik* 5+7=12 is the only arithmetical example. In the former case, then, we could simply have hit upon a random example among others.

The minimal view entailed by hypothesis *i*. conveys some interesting information anyway. Plato picked up a chance number and wrote down a few decompositions of it as a *sum* of numbers less than it. Since explicit reference is made throughout the passage to parts and wholes, the numbers constituting a decomposition are parts of the number to be decomposed (more on the latter two issues in Sect. 2.1 below). Therefore, the presence of the decomposition 6=3+2+1 unambiguously entails that the units composing a number are parts of it, in fact proper parts.[8] The fact is worth a mention: not even the *Elements* are fully explicit on this point, and the definition of proper part of a number (VII.def.3) has often been interpreted as leaving units out, reference being only made to numbers as proper parts.[9] As a matter of fact, in the *Elements* units are parts of numbers, and discussions in recent literature have often forgotten two uncontroversial indications in this sense.[10] First, some subsequent definitions and propositions in the arithmetic books do require that units are proper parts of any number.[11] Second, at least one passage expressly states that "the unit measures every number".[12] In the decompositions "twice three" and "three times two", the reconstruction of a number from one of its proper parts by means of repeated addition is given prominence.[13] The move is obviously reminiscent of the very way proper parts are generated: for instance, the decomposition 6=2+2+2 entails that 2 is a proper part of 6 (cfr. *Elements* VII.def.3).

The historiographical perspectives opened by hypothesis *ii*. are of course more interesting, but the numbers and operations involved in the example are not, as often happens, enough to allow a determination of the underlying mathematical content. An example will clarify the point. It will be argued in what follows that complete numbers are referred to in the Platonic passage, in particular in the decomposition 6=3+2+1. But

[8] A number that exactly measures a number will be caled al "proper part" of it: see the following note, and Sect. 2.1 below, in particular note 30.
[9] Such a reading is indeed fallacious, and is grounded on a misunderstanding of the logic of the definition. Stating that "Part is an ἀριθμός of an ἀριθμός, the lesser of the greater, when it measures out the greater", VII.def.3 refers to a property of ἀριθμοί, and this does not mean that units are to be ruled out as proper parts, but simply that "part" need to be defined for ἀριθμοί only. That units are parts of all ἀριθμοί is the very gist of VII.def.2 (and this shows that VII.def.2 is not mathematically useless), but of course this does not entail that units are ἀριθμοί: a Greek unit can be taken to be a number (in our sense), but it is definitely not an ἀριθμός by the very VII.def.2.
[10] Among the exceptions are e.g. Euclide 1994, p. 252 or Høyrup 2004, in particular the final section.
[11] See *Elements* VII.37 and IX.36 and the very definition VII.def.23 of complete numbers.
[12] *Elements* VIII.6; see Euclides, *Elementa*, vol. II, p. 160.7–8. It is not to be excluded that the clause is a late interpolation; most notably, it is not attested in the mediaeval Latin translations from Arabic.
[13] The two decompositions are *not* to be read as multiplications. Recall that Greek multiplication is much more a shorthand for a repeated addition than our multiplication is (cfr. also *Elements* VII.def.16). On the issue see e.g. Taisbak 2002.

the very same decomposition could be construed as carrying a very different reference. Special properties of certain triples of numbers had in fact been seriously investigated in antiquity, as we know from late sources.[14] As Iamblichus teaches us, take three consecutive numbers, the last of which is divisible by 3, and add them. Add the units in the result and as many units as there are tens, and as many units as there are hundreds, etc. – in our decimal notation this amounts to adding the digits of the result. The final result will always be 6 (the summation must be repeated until a number less than 10 is obtained), we are told without proof. The triple 1, 2, 3 is thus the πυθμήν of all triples of consecutive numbers with the above property, and 6 is the πυθμήν of all numbers obtained as sums of such triples:[15] a πυθμήν is in fact the smallest number, or set of numbers, or ratio, to which a certain property applies; the concept was already well–developed in the times of Plato.[16] One could read Plato's passage as a covered allusion to such a property rather than to complete numbers. To decide between the two interpretations could be impossible, unless further clues come from the context.

A serious problem lies however in locating a threshold above which the "context" provides an amount of data enough to force some precise interpretation. Moreover, "context" is a vague concept and is clearly interpretation–dependent, if only for the fact that each modern interpreter will deem some aspect or other of "the context" as irrelevant to the passage at issue. Nevertheless, an analysis of the context cannot be dispensed with in our case, since the numbers displayed in the passage do not appear to disclose at once a non–trivial mathematical meaning of it. Several layers of what could be broadly termed "context relevant to the passage" can be envisaged: 1) the philosophical context in the dialogue; 2) the mathematical examples in the *Theaetetus*; 3) Plato's stylistic strategies; 4) arithmetic in Plato's times. Let us briefly investigate these points in succession.

2.1. The philosophical context

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[14] Iamblichus, *In Nicomachi arithmeticam introductionem liber*, pp. 103.10–104.13 (Pistelli). The context is an explanation of the concept of arithmetic mean. The triple 1, 2, 3 is just the lowest such mean.

[15] Iamblichus does not speak expressly of πυθμήν in this instance, but he was of course familiar both with the concept and the term (cfr. the *Index verborum* in Pistelli's edition, p. 182b *sub voce*).

[16] Such a meaning is at issue in *Respublica* 546C1–3, and see also Speusippus' fragment analysed below. Referring to the property described in the text, Iamblichus describes the hexad as "image–shaping and element of those [sums of triples] that come after it" (p. 104.2–3 Pistelli). The term πυθμένες is encountered in Hippolytus, *Refutatio omnium haeresium*, IV.14 to denote numbers obtained by adding the numerical values of the letters in Greek proper names until a number less than 10 is obtained. The

Three definitions of knowledge are propounded in the *Theaetetus*.[17] The third, and more sophisticated, definition is first presented by Theaetetus (201C–D) in the form of a definition: knowledge is true judgement with an account (λόγος). Theaetetus' reference is immediately clarified by the doctrine expounded by Socrates in his dream (201E–202C). (Theaetetus admits his inability to provide a fuller account of the reported definition). A major point of the doctrine in the dream is a clear–cut statement about the knowability of elements and complexes: the former are unknowable, the latter knowable. The discussion about this point will take a substantial part of what follows, and will be conducted on the example of letters and syllables, taken as paradigmatic instances of elements and complexes respectively.[18] Socrates declares in fact that he is not satisfied with the doctrine just expounded, and proceeds (202D–206C) to offer two confutations of it. The first confutation (202D–205D) takes the form of a dilemma concerning letters and syllables: Socrates shows that, contrary to what is asserted in the doctrine of the dream, syllables are either as knowable as letters or as unknowable as them. Supposing first that a syllable is the same as all its letters, Socrates quickly concludes to a contradiction with the doctrine of differential knowability of elements and complexes (202E–203D).[19] Second, Socrates envisages the alternative possibility, namely that "the complex be a single form resulting from the combination of the several elements when they fit together; and let this hold both of language and of things in general", and infers that "it must have no parts" (204A). Theaetetus reacts with surprise to such a claim, and the passage quoted *in extenso* at the beginning of this paper follows: Socrates tries to convince Theaetetus using a lenghty and elaborated argument. Theaetetus organizes some resistence to Socrates' pressing reasoning, but eventually he gives his assent, even if somehow recalcitrantly.[20] The upshot of the discussion, once Theaetetus' resistence is defeated, is that "both the whole and all of it are all the parts", whence the claim at 204A just quoted follows. The dialogue returns then (205B–D) to

calculus of πυθμένες was thoroughly investigated by Apollonius, as we know from Pappus, *Collectio* II.1–27 (pp. 3–29 Hultsch).

[17] The following discussion owes very much to the thorough analyses in Burnyeat 1990, pp. 191–218 and Harte 2002, pp. 32–47 and 144–150.

[18] The same Greek word is used for "letter" and "element" (στοιχεῖον); another word for both "complex" and "syllable" (συλλαβή).

[19] Actually, only the assumption that elements are unknowable is reduced to contradiction, since they are shown to be as knowable as syllables.

[20] Read the answers at 204D3, E4, E10, 205E8, and compare them with the final answer to the second confutation (206B12). Socrates acknowledges the tenacity of Theaetetus' defence at 205A1.

the example of letters and syllables and recapitulates the preceding argument, concluding that, since the only things that can be parts of a syllable are the letters, syllables and letters must be both knowable (in case a syllable has parts) or unknowable (in case it has not) at the same time. It is clear that this confutation of the doctrine in the dream depends on one crucial assumption, actually made explicit at several places:[21] a whole is the same as all its parts. Surprisingly enough, at 206A–C a second confutation is offered of the doctrine of differential knowability, and it is to be taken (at least according to Socrates,[22] who appears thus to cast some doubts on the elaborate argument developed in the first confutation) as the real one; the argument comes from the experience everyone has had in learning the basic elements of language or of an art. The second confutation appears to be at variance with the first, in that it shows that elements are more knowable than complexes. Three proposals follow (206C–210B) about the meaning of the term "account" in the definition of knowledge at issue. Only the last proposal (208C–210B) survives Socrates' criticisms, but leads to a circular definition of knowledge. During the discussion of the second proposal (namely that an account of a complex is a complete list of its elements) the dialectics of parts and wholes surfaces again, through the example of the wagon (207A–C) and a resumption of the instantiation by letters and syllables (207D–208A).

This account of the last portion (201C–210D) of the *Theaetetus* gives a rough and partial idea of what is at issue there, but it is enough for our purposes. The mathematical example at 204B–C serves as a first step in the lenghty argument (entirely quoted at the beginning of the paper) devised to convince Theaetetus that "all the parts [of a complex] are all of it", an ontological stance that is crucial to the whole last portion of the dialogue.[23] The mathematical example is thus located at a crucial point of the entire dialogue, in a context that unambiguously refers to the philosophical issues raised by the relationships between a whole and its parts.[24] Since these relationships have also direct mathematical connotations, it is natural to take them as the appropriate

[21] Cfr. 204A7, E3–5, 205A9, D9.
[22] Cfr. 206A1–3.
[23] Even if, as Harte 2002, p. 43, remarks, Plato seems to suggest that it is hazardous to widen the identification to the claim that "a whole is the same as all its parts". The crucial move is thus the identification of a whole with "all of it": unlike the former, the latter expression entails that some division into parts has occurred. See Harte 2002 for a detailed study of the parts–whole dialectics in Plato.
[24] Interestingly enough, the word μέρος is not mentioned when the several decompositions of 6 are listed and briefly discussed. Yet it is an often repeated key word in what immediately precedes and follows.

technical context where to look for the mathematical background of the example at 204B.

Such an identification involves at least three problems. The first problem is to establish the sense in which a number *is* its parts. This point is settled by observing that in numbers sameness is replaced by equality.[25] The conception of numbers as collections of units entails in fact that there are arbitrarily many instances of e.g. a sextet:[26] since there are arbitrarily many different units there are actually arbitrarily many different sextets. Relations between numbers (as e.g. equinumerosity) are thus always established between different instances of them. Saying that "a number *is* its parts" must therefore mean that "a number *is equal to* its parts".

The second problem is the way in which those parts are composed to give the whole. It is clear from some of the decompositions displayed in the Platonic passage that the intended meaning is that the parts are to be *added*, even if this is not expressly said. The practice in later mathematical treatises, most notably in the *Elements*, was exactly that of implying the addition: a case in point is just VII.def.23: "A complete number is the one that is equal to its parts". When an equality concerning the sum of several homogeneous items was at issue, even in a non–definitory context, the rule was that of implying the addition. When the single items were instead to be taken severally, a suitable expression was used, as e.g. "respectively". An example among hundreds is the enunciation of *Elements* I.4.

The third, and substantial, problem is what the parts of numbers are. A first possibility: at 204B–C "part" is not to be taken in any technical meaning. This is to be excluded given the example at issue, even if it is clear that Plato is playing with the technical and non–technical meanings of the term (the latter is of course required outside the strictly arithmetical example, e.g. when the army or the syllables are at issue). After all, paraphrasing Socrates' reasoning at 205B, one could wonder what the parts of a number are if not numbers (or units).

The ancient conception of numbers as collections of units entails that each number is contained in those that follow it, and hence is a part of them in a well–defined

[25] That this was the case is confirmed by Aristotle's remark that "[i]t is said rightly that a number is the same, the one of the sheeps or of the dogs, if both are equal, but a decad is not the same and decads are not the same, in the same way as triangles are not the same, the equilateral and the scalene" (*Physica* Δ 14, 224a2–4). It is interesting to notice that equal ratios are instead invariably said to be "the same".
[26] The discussion in Burnyeat 1990, pp. 205–207, focusses precisely on this point.

sense.[27] The list 1, 2, 3, 4, 5, 6 in the mathematical example at 204B–C really enumerates all parts of a sextet, hinting at the very way a sextet is arrived at: it is composed of (ἐκ) the numbers preceding it. This we could call the weak technical meaning of "part" in arithmetic. On the other hand, there is a strong technical meaning of "part", namely the one codified in *Elements* VII.def.3: "Part is an ἀριθμός of an ἀριθμός, the lesser of the greater, when it measures out [καταμετρῇ] the greater". (In what follows, the term "proper part" shall be consistently used when this will be the intended meaning.) The division of the canon described at *Timaeus* 35C–36B (both forms μέρος and μορίον are attested) and the well–known fragment of Archytas concerning the three basic means attest that such an usage is definitely earlier than the *Elements*.[28] That there was early awareness of the ambiguities entailed by the coexistence of the weak and the strong technical meaning of "part" is clearly displayed by the entry devoted to the concept of part in Aristotle's philosophical lexicon in the *Metaphysica*: "A part is called, in one way, that into which a quantity is divided in a way whatever (for what is subtracted from a quantity *qua* quantity is always called a part of it, e.g. the duet is called a part of the trio in some way), in another way, only those among such parts which measure out [τὰ καταμετροῦντα]; this is why the duet is called a part of the trio in a way, but in a way not".[29] As *Elements* VII.def.3 just above attests, the term transcribed in Greek is a *terminus technicus* in Greek mathematics, most notably in the arithmetic books of the *Elements*: the preposition κατά is prefixed to μετρεῖν exactly when the measuring quantity, after repeated subtraction, exhausts without residue the measured one. Both occurrences of "part" in the passage are thus to be referred to an arithmetical context, so that we are entitled to conclude that an ambiguous meaning of "part" was already present in the mathematical terminology of the middle of the fourth century. As a result of this ambiguity, a seemingly transparent sentence as "the whole is the sum of its parts" runs into difficulties even in a technical context. Apparently to get rid of such difficulties, in *Elements* VII.def.4 a definition of μέρη, "parts" in the plural, is proposed. Whereas a number is "part" of a number exactly when measures it, "parts" of a number is a number less than it but not measuring it: the

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[27] Cfr. *Elements* VII.def.2 and Aristotle, *Metaphysica* Z 13, 1039a12, I 1, 1053a30.

[28] Porphyrius, *In Ptolemaei Harmonica commentarium*, p. 93.6–17 (Düring) (= fr. DK 47 B 2). Cfr. also Speusippus' remarks that the decad contains an equal number of prime and composite numbers ([Iamblichus], *Theologoumena arithmeticae*, p. 83.14–19), a passage to which we shall return in Sect. 2.4 below. A quick check in the TLG_E will show that expressions such as e.g. τὰ δύο μέρη (in the technical meaning of "two thirds") are attested in pre–Platonic orators and historians.

[29] *Metaphysica* Δ 25, 1023b12–17.

weak technical meaning is differentiated from the strong one by passing to the plural. The latter sense of "parts" cannot be explained by the fact that "parts" are made up of a plurality of proper parts, since this is true in general only if the proper parts are units. If this lexical convention rules out ambiguities in the use of "part" in the singular, the ambiguities remain when several parts are referred to, since it is not clear whether a plurality of proper parts is at issue or a plurality of "parts" or both.[30] Unfortunately, the use of "parts" in the present meaning in the mathematical literature is restricted to the arithmetic books of the *Elements*: it is likely that the inconvenience of that lexical choice was detected, and the term fell out of usage.

The preceding discussion rules out the possibility that in the example at 204B–C Plato is referring solely to the weak technical meaning of "part"[31] or, assuming that this could have been the case, to the Euclidean sense of "parts".[32] On the contrary, Plato was in position to refer to *proper parts* of a number. Moreover, as seen above, he definitely adds those parts, and we can safely assume that a number *is* its parts in the sense that it *is equal* to them. But even after such a sharpening of the lexical apparatus, a problem remains with e.g. the claim at 204D1–2: "in all things made up of number, at any rate, by 'all of it' and 'all of them' we mean the same thing" – the problem is that the claim is false. Not all numbers are the sum of their own proper parts, even if some numbers are, and when this happens mathematicians are used, at least from *Elements* VII.def.23 on, to call them "complete numbers": τέλειοι ἀριθμοί. It is at this point of the dialogue that Theaetetus entered Socrates' trap: that "the whole is the sum of its parts" is false even in numbers.[33] Theaetetus is wrong in his giving assent to that statement, and he is really catching one of those "pieces of ignorance" just evoked in the simile of the aviary.[34] It is not necessary to develop the notion of complete number to realize that e.g. an octet is not the sum of its proper parts. But Plato singled out just a sextet in the example, and the sextet happens to be the sum of its proper parts: 6=3+2+1, as in the last decomposition in the list. So Theaetetus is in general grossly mistaken, in that not every

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[30] In modern scholarly literature the problem is somewhat hidden by the usage of qualifyiers of the word "part"; I shall use "proper parts" as the plural of a part. I shall use in the sequel "aliquot parts" for what are our unit fractions.

[31] In Burnyeat 1990 and Harte 2002 only this possibility is envisaged.

[32] Recall also that the argument aims at convincing Theaetetus, a valuable albeit young mathematician: he could easily have remarked that, in either hypotheses, no number could ever be the same as *all* its parts.

[33] By the way, this seems to support the Platonic characterization of the young Theaetetus as relying on true judgement falling short of knowledge. See on this Burnyeat 1990, pp. 129 ff. But Theaetetus is driven to error by Socrates' fallacious contention that "all of it is all of them". This is a further clue that Plato is subtly suggesting that there is something wrong in Socrates' argument.

number is the sum of its proper parts, but in the specific case of a sextet he is right, in that it *is* the sum of its proper parts. Theaetetus is driven to error by its own mathematical abilities.

We could also try to enlarge the range of the philosophical issues that can be brought to bear on the mathematical example. The dialogue is both an investigation of what knowledge is and (hence) an inquiry about what a good definition of knowledge should be. Therefore a proper context to the mathematical examples in the dialogue is the philosophical assessment of the meaning and role of definitions in mathematics. This point has been strongly made by M. Burnyeat with reference to the philosophical significance of the passage at 147D–148B. In his view, the lesson to be drawn from 147D–148B should be that definitions are better intended as provisional points of any inquiry and not as its goal, even before this point is made clear by the rest of the dialogue.[35] That something concerning definitions as starting points is at issue in 204A–205A too is somewhat corroborated by the fact that the arithmetical example is introduced to explain a point concerning elements (στοιχεῖα) and complexes. It is well known that στοιχεῖον was standard mathemathical terminology in the Academy at least by the times of Menaechmus, who cared to distinguish two senses in which something can be said to be an element of something other.[36] Among elements there are definitions:[37] the arithmetical example at 204B–C could refer to a *definition* of a precise mathematical notion.

Collecting the remarks above it is natural to read one among Plato's decompositions, namely 6=3+2+1, as an allusion both to the definition of complete numbers and to the sextet as a complete number, written as the sum of its proper parts. But this proposal leaves some questions open: who was the inventor of complete numbers, and why is Plato not explicit on the whole issue? Some clues come from other diffuse "contexts" relevant to our passage, the first of which is constituted by the several mathematical examples in the dialogue.

2.2. The mathematical examples in the Theaetetus

[34] 196C–199D. Recall also that the paradigmatic mistake in the simile of the aviary is a mathematical one.
[35] See Burnyeat 1978, in particular pp. 509–512.
[36] Proclus, *In primum Euclidis Elementorum librum commentarii*, pp. 72.23–73.14 (Friedlein). Menaechmus' elaboration is not explicitly said by Proclus to refer to mathematical issues, but such an origin is very likely.

The number of mathematical examples presented in the dialogue can be easily justified, besides Plato's well–known predilection for such examples: Theaetetus was a distinguished mathematician. The ancient sources, however scanty, are unanimous on this. Of course, the main testimony is the *Theaetetus* itself, in particular at 147D–148B, if the value of the passage as a historical record is accepted. The young Theaetetus is there reported to have proceeded to a first classification of lines that are commensurable or incommensurable with a preassigned one. As a matter of fact, the classification amounts to a *definition* laying down the basis for any systematic investigation on irrational lines.[38] In our perspective it is important to stress that the definition rests upon number–theoretic concepts and is expressed in a number–theoretic idiom. The only uncertainty is about the date of such an achievement, but it is very likely that Plato's dramatic exigencies forced him to conspicuously predate the first conception of the definition.

In Pappus' *Commentary on Book X of Euclid's Elements*, Theaetetus is reported to have introduced and named the three basic kinds of irrational lines (medial, binomial, and apotome), linking them to the three basic means (geometric, arithmetic, and harmonic respectively).[39] The testimony of Pappus, though expressly referring to the *Theaetetus*, appears to carry additional and independent information, whose source is asserted to be Eudemus. Proclus mentions Theaetetus in tandem with Archytas, and asserts that by them "the theorems were increased and an advance was made towards a more scientific grouping".[40] Theaetetus' results are then said to have been perfected by Hermotimus and eventually by Euclid. Proclus offers no clue about the character of Theaetetus' fields of activity. The pairing with Archytas, whose main contributions appear to be of a number–theoretical nature,[41] cannot be underestimated, since the connection was very likely already explicit in Proclus' sources.

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[37] This seems uncontroversial already for Plato: see e.g. the discussion in Mueller 1991.

[38] Definition and terminology do not match with the corresponding ones in the *Elements*, and this has somewhat bewildered many commentators. See the remarks in Vitrac (forthcoming).

[39] Pappus, *Commentary on Book X of Euclid's Elements* I.1, p. 63 (Junge–Thomson). See also II.17, p. 138. The first occurrence of the denominations "apotome" and "binomial" is in (Pseudo–)Aristoteles, *De lineis insecabilibus* 968b20. This small treatise is a product of the Peripatetic school. A work with the same title is included also in the list of Theophrastus' writings (see e.g. Diogenes Laertius, *Vitae philosophorum* V.42). Therefore, it is reasonable to assume that it has been composed before the *Elements*.

[40] Proclus, *In primum Euclidis Elementorum librum commentarii*, p. 66.15–18 (Friedlein).

[41] Most notably the theory of means and its application to canonic division. B. L. van der Waerden, on highly questionable grounds, propounds to ascribe book VIII of the *Elements* to him (see van der Waerden 1954, pp. 152–155). Knorr 1975, pp. 212–225, substantially endorses such a position.

Since much of what is required for the theory of irrational lines in book X of the *Elements* belongs in the arithmetic books, H. Zeuthen ascribed the bulk of books VII and VIII to Theaetetus.[42] A view like Zeuthen's is advocated by W. R. Knorr: he assigns to Theaetetus, first, a body of theorems that was organized in strict deductive sequence and that eventually surfaced as the main portion of *Elements* VII;[43] second, the foundations and further elaboration of a sequence of theorems very similar to what is actually expounded in the initial segment of *Elements* X.[44]

To Theaetetus are also ascribed, though to a variable extent, the discovery, construction, and classification of (some of) the regular solids.[45] The main source of information is the liminar scholium to *Elements* XIII: " […] three of the said 5 figures belong to the Pythagoreans, the cube and the pyramid and the dodecahedron, whereas the octahedron and the icosahedron belong to Theaetetus".[46] Following the first, fundamental study of E. Sachs, W. C. Waterhouse strongly advocates the thesis that the testimony of the scholium is fully reliable[47] and that Theaetetus actually shaped the definition of regular solid implicit in the *Elements*, thereby proving that there are only five of them.[48]

In short, the two main achievements ascribed to Theaetetus in antiquity are of a systematic character; in particular it appears that he was particularly clever in finding fruitful and long–standing definitions. Moreover, even if Theaetetus' range of mathematical interests is wide, we can safely claim that arithmetic constituted the core of his research activity. This is well in keeping with the prominence of arithmetical examples in the *Theaetetus*. A very brief survey will suffice to show this.

The arithmetical emphasis in 147D–148B has been pointed out above, and the 5+7=12 (or 11?) sum at 196A and 199B has already been discussed. At 154C the question is whether it is possible to become bigger in number in any other way than

[42] See Zeuthen 1910, in particular pp. 408, 420–422. Notice that a similar historiographical stance is otherwise deemed as *naïve* when assumed e.g. by Proclus in his ascribing to Thales theorems that are needed to explain some of his practical achievements. See Dicks 1959 on this.
[43] Recall Proclus' assessment quoted just above.
[44] Knorr 1975, pp. 238–244.
[45] A thorough discussion can be found in Sachs 1917.
[46] Euclides, *Elementa*, vol. V.2, scholium 1 in librum XIII, p. 291.1–9 (Heiberg–Stamatis). The entry devoted to Theaetetus in the *Suda* lexicon (θ 93) claims: "Theaetetus […] first described the so–called five solids" (Adler 1928-38).
[47] The gist of the argument is that the discovery of the octahedron – whose construction is very simple – at a later stage can satisfactorily be accounted for only if a systematic perspective has already been developed.
[48] Waterhouse 1972–73; see also the useful discussion in Euclide 2001, pp. 97–104.

being increased.[49] The example involves six dice, that are asserted to be three halves (ἡμιολίους) of four dice, but half of twelve. Technical terms are employed here, even if they are not required by the context. At 190B Socrates claims that not even in his sleep has Theaetetus gone so far as to say to himself "No doubt the odd is even". At 198A–C the arithmetic τέχνη is a "science concerning every even and odd", whereas ἀριθμεῖν is defined as "inquiring how big a certain number happens to be".[50] To these passages K. Gaiser's suggestion could be added,[51] who regards the text at 202B (τὰ μὲν στοιχεῖα ἄλογα ... τὰς δὲ συλλαβὰς γνωστὰς τε καὶ ῥητάς) as a pun alluding to the distinction between ἄλογος and ῥητός, a basic one in the later theory of irrationals.[52]

The dialogue appears thus to stress arithmetical issues, perhaps suggesting that they were a prominent feature of Theaetetus' investigations. We are thus naturally led to connect Theaetetus to the invention of complete numbers. The way the example at 204B–C is presented will give further weight to the proposal.

2.3. Plato's stylistic strategies

It is clearly out of place to discuss here the complex stylistic strategies that can be discerned in Plato's dialogues (nor would I be able to do that). I will concentrate my attention on a simple question: why is the example at 204B–C not explicit?

A first possibility is that Plato had intended to present a fully transparent example, at least for those who were in a position to have an informed access to his dialogue. Historians have not been able till now to enucleate the mathematical reference of the passage simply because of the present fragmentary knowledge of the mathematics in Plato's times.[53]

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[49] αὐξηθέν in the text. The verb frequently refers to increase through multiplication (see e.g. *Respublica* 546B–C and 587D9), but here a more generic meaning is very likely at issue.
[50] The same definition is in *Gorgias* 451B3–4.
[51] Gaiser 1968, p. 472. The same remark is in Burnyeat 1978, note 78 on p. 510.
[52] See e.g. *Elements* X.def.3. For ῥητός as opposed to ἄλογος in technical contexts before Euclid see e.g. Plato, *Respublica* 534D5 and (Pseudo–)Aristoteles, *De lineis insecabilibus* 968b19. The same opposition is expressed in terms of the contraposition between ῥητός and ἀρρήτος in the geometrical–number passage at *Respublica* 546C4–5 (the latter term does not appear in the *Elements*). Therefore, we are not entitled to conclude that the very precise Euclidean meaning of the terms was already in usage in Plato's times: their several occurrences in the *Theaetetus*, as well as many of those recorded in Fowler 1999, pp. 191–193, are unlikely to refer to mathematical issues.
[53] Ancient commentaries are of no help in the interpretation of the passage. In Greene's *Scholia Platonica* no scholia to the *Theaetetus* are reported ranging from 201B to 206D. The anonymous commentary to the *Theaetetus* ends with lemma 157E4–158A2 and comments thereon.

A second possibility is that the example had been expressed in obscure terms for dramatic or other reasons. We could refer to Plato's well–known habit of expressing *all* mathematical examples in a non–transparent fashion. Just a few examples. The generation of the soul in *Timaeus* 35C–36B amounts to a detailed description of an early canonic division.[54] The mysterious geometrical number in *Respublica* 546B–D determines the cycles of fertility in human beings. Interestingly enough, in the same passage mention is made (546B4) of some "complete" (τέλειος) number ruling the same affairs for celestial entities. In *Leges* 737E–738A, 5040 is mentioned as the ideal number of households in a state; it is also asserted that 5040 admits as proper parts all number from one to ten, and that the number of its proper parts is sixty less one. In *Leges* 771B–D it is observed that the same number divided by 12 yields exactly twenty–one times twenty, and the latter – the number of tribes – can be divided by twelve again. Further, 5040 is said to admit all parts up to twelve, eleven excluded, even if it is easy to remedy to this by subtracting two households. In *Respublica* 587C–D the tyrant is declared to be up to (3×3)×(3×3)×(3×3) (=729) times removed from pleasure than a wise man – and this is clear to the calculator. Other examples could be adduced, but the proposal of a generalized strategy of making the mathematical references obscure is half–way to begging the question, and simply shifts the problem.

Third possibility: Plato purposedly gives the example at 204B–C a cryptic expression *as a mathematical reference.* This view is compatible with the hypothesis that Plato is alluding to a recent mathematical achievement, and that he wants to keep it transparent to a very restricted *élite* only. The opacity of the mathematical reference could for instance be explained by assuming that Theaetetus did not carry any investigations on the subject. Making the mathematical reference explicit could instead have induced the reader into wrongly believing that the subject at issue had to be associated with Theaetetus. This being not the case, Plato preferred to make the reference opaque. If instead Theaetetus was the inventor of complete numbers, the example could be read as a homage *in extremis*, celebrating one of his main results in a very private form. In this perspective, a kind of progression towards obscurity should be read in the sequence of the mathematical passages of the *Theaetetus*, ranging from the

[54] The canonic division in the *Timaeus* is commonly ascribed to Philolaus on the grounds of a testimony in Nicomachus' *Encheiridion* (see pp. 252.17–253.3 (Jan) or fragment DK 44 B 6). See e.g. Burkert 1972, in particular pp. 386–400, whose position is endorsed in Huffman 1993. However, several scholars regard fragment DK 44 B 6 as a late forgery inspired by the passage in the *Timaeus*. See, first and foremost, Tannery 1904.

very explicit example in 147D–148B to the cryptic one in 204A–205A through a series of covered allusions to arithmetic. In particular, the allusion to a "complete (τελέως) arithmetician" at 198B9 would constitute a bridge between the former example and the latter, maybe suggesting that the latter too has to be taken seriously.[55] Of course, Theaetetus himself may have elaborated preexisting material, sorting a well–thought and fruitful definition out of some mathematical technique.

All of the above leads me to surmise that Theaetetus himself was the inventor of the notion of complete number, that Plato is referring to this fact in *Theaetetus* 204A–205A, and that the way he shapes the reference is a masterpiece of irony and sophistication.[56] We could even construe Plato's opacity as partly intended to hide a criticism to a definition carrying a certain amount of ambiguity.[57] Given the problems with the terminology of parts explained in Sect. 2.1 above, the definition of complete numbers is in fact ambiguous: of course, if "part" is to be taken in the weak technical meaning, *no* number will be the same as its parts. Plato could therefore be read as, among other things, criticizing early occurrences of such an ambiguous usage: a quartet is not a proper part of a sextet, nevertheless 6=4+2 is proposed as an acceptable decomposition into parts.[58] Both Plato with this decomposition and Euclid with the introduction of a technical meaning of "parts" appear thus to criticize the same source of ambiguities: the fact that any number less than another can be taken to be a part of it. Moreover, to criticize a definition by referring to it through examples[59] would be in keeping with the tension within the dialogue between definitions proper and definitions through examples. Perhaps Plato shaped the first argument against the theory in the dream around a piece of mathematical knowledge (some numbers are the sum of their parts, others are not).

2.4. Some ancient arithmetic

[55] The Greek term occurs, near the passage we are discussing, at 202C4 and four times in 206A–C.
[56] Recall that Theaetetus is mistaken in giving assent to the claim that any number is the sum of its parts.
[57] There is no need to suppose that the term "complete" was introduced from the very outset. We shall see below that it is very likely a late choice, maybe Euclid's. But recall the "complete mathematician".
[58] Remark that the decomposition 6=1+5, a part (VII.def.2) + parts (VII.def.4), is absent. It is difficult to attach to such an absence a meaning.
[59] Notice also that knowledge is here assumed to be true judgment with λόγος (account), and the whole dialogue raises criticisms to three λόγοι (definitions) of knowledge. Could a further, more specific λόγος, be criticized at 204 B–E?

We are thus led to trace the residual tracks of complete numbers in the Greek mathematical corpus. It is disappointing how scanty the evidence is. Let us briefly review it.

In the well–known fragment from Speusippus in the *Theologoumena arithmeticae* the decad is repeatedly termed τέλειος.[60] Aristotle confirms that the adjective applied to the decad belonged to the standard terminology of Pythagorean tradition,[61] even if Speusippus is not flatly reporting Pythagorean material, as the author of the *Theologoumena* appears to suggest.[62] In Speusippus' perspective, the decad has several properties in virtue of which it is τέλειος, some properties being peculiar to it, others being shared by other numbers.[63] None of the properties fits the definition of complete numbers we find in the *Elements*, but it could be more than coincidence that the decad is termed "complete" because it is the *sum* of a unit and of the first three ἀριθμοί.[64] To the Pythagoreans, τέλειος really means "complete" in a generic sense, as is clear from Aristotle: "[The Pythagoreans] proceed as though number up to the decad were complete. At any rate they generate the things that follow – e.g. the void, proportion, the odd, and others of this sort – within the decad".[65] The remark is clearly intended to throw very bad light on the reported conception, but the terminological point is clear enough and untouched by Aristotle's negative bias. We are thus not entitled to view the ancient Pythagoreans' use of τέλειος as referring to a definition of complete numbers in some technical sense, even less in a number–theoretic one. The term simply stressed the completeness of the decad with reference to a certain number of criteria, let them be arithmetical, numerological, philosophical. Of course, negative terminological evidence does not entail that before Euclid no inquiry had been pursued about numbers that are equal to their own parts; it only gives weight to the contention that the name τέλειος for such numbers is the result of a stipulation that occurred after

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60 [Iamblichus], *Theologoumena arithmeticae*, pp. 82.10–85.23 (De Falco) (= fr. 28 Tarán), most notably at p. 83.6–11. For a thorough analysis of the fragment see Tarán 1981, pp. 257–298.

61 *Metaphysica* A 5, 986a8–9: "since the decad appears to be τέλειον, and to contain every nature of numbers […]".

62 Several mathematical conceptions in the fragment, e.g. that the unit is a number, in fact the first prime number, are at variance with Pythagorean doctrines.

63 In one case among the latter the decad is singled out as being the πυθμήν of all numbers having the property of containing an equal number of prime and composite numbers (*Theologoumena arithmeticae*, p. 83.14–19 (De Falco)).

64 [Iamblichus], *Theologoumena arithmeticae*, p. 84.10–14 (De Falco).

65 *Metaphysica* M 8, 1084a31–34. Annas' translation, slighty modified. Annas' notes to the passage are quite useful.

Aristotle (perhaps just as a reaction to the non–technical connotations of Pythagorean terminology).[66]

After Euclid, to whom we shall turn in a moment, the issue of τέλειοι numbers was resumed by the Neo–Pythagoreans.[67] Theon Smyrnaeus reports[68] the definition of complete numbers and the enunciation of the sufficient condition for a number to be complete stated in *Elements* IX.36 (see below).[69] 6 and 28 are provided as examples of complete numbers. After defining overcomplete and defective numbers, Theon points out that also 10 is termed τέλειος by the Pythagoreans "in accordance with another account".[70] Finally, he asserts that also 3 is so named.[71] Both Nicomachus and Iamblichus provide further information, even if of limited value. Nicomachus[72] first defines overcomplete, defective, and complete numbers as particular species of *even* numbers. He then lists the first four complete numbers, namely 6, 28, 496, 8128, describing the way of their generation according to the Euclidean prescription, and makes three claims. First, there is only one complete number in the units, one in the tens, one in the hundreds, one in the thousands (and this is plain truth). Second, complete numbers will always end in 6 or 8 alternately (false). Third, the sufficient condition for a number to be complete is also necessary (no proof is provided).[73] Iamblichus repeats all of this, upgrading the first claim to a "natural law", and guesses that there is only one complete number in every successive order of myriads,[74] He ends

[66] That is to say that the choice was purposedly polemical: it marked the older meaning of "complete" as unserviceable in mathematics, thereby trying to sweep it out of the technical field. The attempt was unsuccessful, as is clear from the later evidence discussed below.
[67] Later discussions as e.g. the one in *Davidis Prolegomena et in Porphyrii Isagogen commentarium*, p. 22.18–35 (Busse) are of no interest.
[68] Theon Smyrnaeus, *Expositio rerum ad legendum Platonem utilium*, pp. 45.9–46.19 (Hiller).
[69] Theon's statements fit the enunciations of *Elements* VII.def.23 and IX.36, respectively, even if the these propositions are not explicitly referred to.
[70] The alternative account is expounded (p. 106.7–11 Hiller) in the section devoted to the properties of the first ten numbers, and coincides with the one that can be extracted from the passages in *Metaphysica* A 5 and M 8 referred to above. Among the listed properties of 6 is the one of being a complete number in Euclidean sense.
[71] Cfr. also Aristoteles, *De Cœlo*, A 1, 268a10–11, where it is asserted that the Pythagoreans regarded 3 as the number defining both τὸ πᾶν and τὰ πάντα. In the *Theologoumena arithmeticae*, 9 is said to be complete "in that it is generated from 3 complete" (p. 78.16 De Falco).
[72] Nicomachus, *Introductio arithmetica* I.16, pp. 36.6–44.7 (Hoche).
[73] True (Euler proved this), but only recalling that from the very outset Nicomachus defined complete numbers as a subspecies of even numbers.
[74] Iamblichus, *In Nicomachi arithmeticam introductionem liber*, pp. 31.22–34.26 (Pistelli). The (false) claim is very interesting, in that attests an original conception of what we would call "orders of magnitude". To Iamblichus, it is natural to pass from thousands to the first order of myriads (i.e. the numbers ranging from one myriad to one myriad of myriads), not from thousands to myriads (i.e. the numbers ranging from one myriad to ten myriads). The myriad is thus taken as the unit of the successive order of numbers. Clearly such a conception is connected with Apollonius' tetradic notation for large numbers expounded in book II of Pappus' *Collectio* (pp. 3–29 Hultsch). A different conception underlies

his excursus by defining friendly numbers, and ascribes their discovery to Pythagoras.[75] This is the only piece of evidence that investigations about the issue were carried before Plato, but the ascription looks very much as a retrospective arithmetical interpretation of a standard anecdote, and its value as a testimony is nil. The scholia to *Elements* IX.36 appear to depend on the Neo–Pythagorean tradition and provide no further information.[76]

Since antiquity, a residual line of Euphorion's[77] collection of poems Μοψοπία has been interpreted as referring to complete numbers.[78] The line is quoted to this effect in a digression on complete numbers contained in an anonymous commentary to Porphyry's *Isagoge*. The line tentatively reads "σφοῖσιν ἴσοι μελέεσσι, τὸ καὶ καλέουσι τελείους".[79] A translation could be "equal to his (their) limbs, so that they are called complete". Structure and mathematical content of the digression in the commentary strictly follow analogous discussions in the Neo–Pythagorean tradition, especially in Nicomachus, but the quotation from Euphorion is not attested in any other sources.[80] The anonymous commentator's quotation of Euphorion's line cannot be taken as entailing that the former had access to a text larger than the line itself. Actually, the most likely source of the line is an earlier compilation, where the connection with complete numbers was already made explicit. Again, this does not mean that there were any allusions to complete numbers in Euphorion's original poem too: we are used to such *a posteriori* over–interpretations in late compilators. Unfortunately we have no indications about the context in which the surviving line was inserted, but other fragments of Euphorion suggest that the Μοψοπία dealt, among other things, with Dionysus' death by dismemberment.[81] The god's limbs were then reassembled before his

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the names δευτερωδουμένη, τριωδουμένη … μονάς for 10, 100, etc. (the notion is expressly ascribed to the Pythagoreans e.g. in Iamblichus, *In Nicomachi arithmeticam introductionem liber,* pp. 88.21 ff., 103.16 ff. (Pistelli)).

[75] Iamblichus, *In Nicomachi arithmeticam introductionem liber*, p. 35.1–7 (Pistelli).

[76] Euclides, *Elementa*, vol. V.2, scholia 44–47 in librum IX, pp. 81.14–28 (Heiberg–Stamatis).

[77] Euphorion was a a poet from Euboea, active in Athens and Antioch in mid–third–century B.C.. See e.g. the entry on Euphorion (ε 3801) in the *Suda* lexicon (Adler 1928-38).

[78] See most recently Lightfoot 1998.

[79] The fragment was discovered by G. L. Westerink and brought to scholarly attention in Westerink 1960. The whole commentary has been edited in Westerink 1967; see in particular p. 8.20. The line is badly corrupt, and the reported reading is Westerink's restoration. The text carried by the (two) manuscripts is σφίσιν οἷσι μελέσσι τῶι καὶ καλέονται τέλειοι.

[80] The content of the commentary overlaps also with the content of the commentaries of David and Elias, and the agreement is often *verbatim*. The anonymous author has been approximately dated by Westerink from the end of the sixth century to the beginning of the eight.

[81] The suggestion was first made in Barigazzi 1963, in particular pp. 447–8. A poem under the title *Dionysus* is attested for Euphorion, and it has been suggested that it was contained in the Μοψοπία.

burial in Delphi.[82] It is disappointing that seven (plus the undivided heart) is the transmitted number of pieces into wich Dionysus was divided. Any reference to complete numbers in their technical sense in Euphorion's poems rests thus on a conjectural reconstruction of a handful of fragments: it may well result from lexical coincidences.[83] At any rate, the uncertainty in the dates of both Euphorion and – most notably – Euclid would heavily downplay even the value of a sounder testimony.

Let us come to the *Elements*: VII.def.23 states that "a τέλειος number is that which is equal to its parts". The subject is kept in lethargy until proposition IX.36, the very last one of the arithmetic books of the *Elements*. The theorem states a sufficient condition for a number to be complete in the sense of the just stated definition, and it is a true feat of mathematical insight: "If as many number as we please are set out in duplicate proportion in succession from a unit, until the entire sum is prime, and the sum total multiplied into the last <term> makes some <number>, the result will be complete".[84] Two remarks support the view that an authorial strategy has been permitted to surface in the organization of the treatment of complete numbers in the *Elements*. First, the evidence so far surveyed, most notably Speusippus' fragment, suggests that the denomination τέλειος for a number equal to the sum of its proper parts could be ascribed to Euclid: C. M. Taisbak suggested to read this as a mathematical pun (such numbers are "complete" in the sense that they are really the sum of their proper parts).[85] Moreover, IX.36 ends the arithmetic books of the *Elements* and can properly be taken as their τέλος. Second, *Elements* IX.36 comes just after a sequence (IX.21–34) of astonishingly simple theorems establishing some properties of odd and even numbres. Only the last four propositions in the sequence carry a decent mathematical content, the others being utterly trivial if not empty in their proofs.[86] It is not plausible to regard such a steep climax of mathematical complexity as a consequence of a rather haphazard and careless way of composing the *Elements*.[87] I would rather consider it as a little *coup de théâtre*. Theorem IX.36 comes thus rather out of the blue, and the obvious inference that there must have been some serious elaborations on complete numbers before Euclid was

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[82] One is strongly reminded of the story of the so–called "Horus–eye fractions". In Ritter 2002 it is shown that they are no more than a historiographical myth.

[83] Lightfoot's conclusions are explicit on this.

[84] As said above, the sufficient condition stated in IX.36 is also necessary for even perfect numbers. Nothing is know about the existence of odd perfect numbers. It is not known whether there are infinitely many perfect numbers or not. At present, a few tens of perfect numbers are known.

[85] See Taisbak 1976.

[86] On the character of these proofs see e.g. Mueller 1981, pp. 103–106.

[87] This was the opinion of O. Becker, see below.

not supported so far by any evidence. This state of affairs has left room to two weighty reconstructions.

Years ago, Taisbak proposed a rather straightforward reconstruction of the genesis and invention of complete numbers as a consequence of an insightful inspection of the multiplication tables typical of the so–called Egyptian method.[88] The method is applied in several problems in the Rhind mathematical papyrus and can be summmarized thus.[89] Given two numbers to be multiplied, other numbers are set in two columns according to the following procedure. The left column always carries a unit and successive duplications of it. The right column carries first one of the numbers to be multiplied and, under it, successive duplications of it, in correspondence with the numbers set out in the left column. Strokes mark the numbers on the left column that sum up to give the other number to be multiplied,[90] and the final result is obtained by summing the corresponding numbers in the right column. Take for instance the multiplication tables of 3×3 (left) and of 7×7 (right):

/	1	3	/	1	7
/	2	6	/	2	14
summing 3 and 6 yields 9			/	4	28
			summing 7 and 14 and 28 yields 49		

Notice that in both tables all numbers in the left column are marked by strokes. Moreover, the last numbers of the right columns in both tables, namely 6 and 28, are the sum of all the other numbers in the table. It is easy to realize that every figure in the left column has a stroke exactly when the number to be multiplied by itself is a unit less than an even–times–even–only number (a particular case of *Elements* IX.35).[91] It was a later step, Taisbak suggests, to realize that only in some cases among these the last number in the table was measured by the others and only by them, and that this occurs

[88] See Taisbak1976.
[89] On this see most recently Clagett 1999, pp. 113–204.
[90] The method works in that every number can be written as a sum of suitable powers of two and possibly one unit. It is not at all clear whether Egyptian or Greek arithmeticians had developed this argument in the form of a proof, even sketchy.
[91] Egyptian scribes knew of sequences of numbers obtained as successive duplications; a summation procedure, up to a fixed and not generic number of terms, is sketched in problem n. 79 of the Rhind papyrus. It is not clear what such problems are to be taken to entail about an Egyptian acquaintance with summation procedures of geometric progressions. See Clagett 1999, pp. 58–60 and, for a less nuanced assessment, Caveing 1994, vol. 1, pp. 377–378.

when the first number in the right column is prime. Taisbak claims thus that complete numbers "were discovered at the reckoning board". The hard work was of course to set out a proof from these remarks, but it is clear that, in Taisbak's view, invention of complete numbers and proof of the sufficient condition go together.

Independent support to the hypothesis that complete numbers were known and studied before Euclid comes from a well–known reconstruction of O. Becker. He argues that IX.21–34 is a piece of archaic mathematics developing a theory of the odd and the even. The theory had originally been cast in the language of pebbles–arithmetic and was subsequently modified before being included in the *Elements*. Becker shows also that IX.36 can be proved as a consequence of those theorems only, and favours the view that Euclid had conspicuously reworked a pre–existent proof.[92] In Becker's perspective, the final segment of book IX was added, at a relatively late stage and maybe by Euclid himself, to an already established body of theorems. This obviously entails that the whole theory had already a definite shape, albeit possibly a rough one, before Euclid.

Both scenarios (Becker's in a less straightforward way) establish a connection with the so-called Egyptian calculations. It is definitely known that such calculations, performed with the aid of duplication procedures and aliquot parts, were known in Greek mathematics. The denomination "Egyptian method" for these procedures is attested in such late sources as the well–known scholium to *Charmides* 165E[93] or a letter where Psellus explains some terminology implied in Diophantus' *Arithmetica*.[94] The problem lies in the determination of the period and ways of transmission of the Egyptian techniques to Greek–writing mathematicians. In envisaging early borrowings from foreign cultures we are touching on a very difficult historiographic question, since contemporary sources are in most cases absent. Recent and less recent scholarship has more or less cautiously tried to explore such a *terra incognita*. The soundest results come from ancient astronomy, where the Babylonian origin of some parameters and

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92 See Becker 1936 and Mueller 1981, pp. 104–106. The proof is termed "combinatorial" in the latter work. In this very respect, Becker's reconstruction is at variance with Taisbak's, in that the latter appears to entail an original proof of the sufficient condition for complete numbers very similar to the one in the *Elements*.

93 "Parts of it [*scil.* of logistic] are the so–called Hellenic and Egyptian methods in multiplications and divisions, as well as recapitulations and separations of proper parts […]" (p. 115 Greene). The scholium is assigned to Anatolius in Tannery 1887, pp. 48–52. For further discussions see Klein 1968, p. 12 ff. and Caveing 1994, vol. 2, pp. 172–176.

94 Diophantus Alexandrinus, *Opera Omnia*, II, pp. 37.3–39.10 (Tannery), in particular pp. 38.22 and 39.5. In Knorr 1993, note 18 on p. 189, it is proposed that the denomination "Egyptian" is instead to be simply taken as a reference to Diophantus' situation in Alexandria. Knorr's proposal is ruled out by the quick description of the "Egyptian analysis" in the very same letter by Psellus (p. 39.4–9 Tannery).

arithmetical schemes typical of Greek pre–Ptolemaic astronomy is well–established. In particular, Meton's 19–year calendric cycle (late V century B.C.) has been proposed on good grounds to be a development of a similar Babylonian scheme.[95] A. Jones speaks of "a gradual trickle of basic concepts and occasional parameters from about 500 B.C. followed by a sudden flood of detailed information in the second century B.C.".[96] J. Høyrup's researches suggest a continuity between some problems and notions peculiar to Old Babylonian mathematics and Greek subsequent elaborations. Høyrup explains the supposedly ambiguous use of δύναμις in *Theaetetus* 147C–148D (both a square and its side appear to be referred to by the term) by setting it in parallel with the analogous ambiguity in the corresponding Babylonian term *mithartum*. The proposal of direct borrowing can be consistently argued, even if the documentary evidence does not settle the point.[97] More importantly, a whole tradition of Near Eastern practitioners' problems can be followed in its transmission and transformations through Greek metric tradition (possibly also *Elements* book II),[98] Diophantus, and Islamic *al–jabr* up to Renaissance treatises, even if early Greek evidence is lacking.[99]

A continuity in the field of logistic is strongly suggested by the persistence over millennia of a stable body of computational techniques, most notably the consistent and uninterrupted use of aliquot parts in both calculations involving parts and in the presentation of the final results.[100] But, also in this case, the evidence for the crucial period is virtually non–existent, the earliest records for the Hellenic world being a set of demotic papyri of mathematical argument, ranging from the third century B.C. to the second century A.D..[101] The metrical tradition collected in the Heronian corpus gives us further, albeit late, confirmation of such a continuity. More generally, the few reliable testimonies we have about pre–Euclidean Greek mathematics attest for a particular

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95 See Bowen and Goldstein 1988, in particular pp. 48–51. How ancient is the scheme is an open problem; see *ibidem*.

96 Jones 1993, quotation from p. 88.

97 See Høyrup 1990a, pp. 201–202.

98 The similarity of the geometric theorems in *Elements* II to some "cut and paste" problems typical of old Mesopotamian mathematics was the historical basis by which the thesis of Greek "geometrical algebra" was *a posteriori* believed to be supported. See e.g. Neugebauer 1934–36. As is well know, the original idea of a Greek "geometrical algebra" goes back at least to Tannery and was expressed in clear–cut terms in Zeuthen's *Die Lehre von den Kegelschnitten im Altertum*.

99 See e.g. Høyrup 1996 and 1997.

100 See Knorr 1982, most notably pp. 154–160, Fowler 1992, and Vitrac 1992. A very interesting analysis is in Høyrup 1990b, see in particular the final discussion.

101 We are not in position to guess any date for the arithmetical riddles in book XIV of the *Anthologia Græca*.

interest in arithmetic and logistic.[102] At least by the times of Archytas number–theoretic investigations are attested, possibly triggered by harmonic theory and in particular by investigations on the several division of the canon acceptable from a theoretical or an empirical point of view.[103]

A strong case has been made by M. Caveing in favour of a very early conceptualization of complete numbers.[104] Caveing's proposal is the upshot of a long–standing tradition, that links the singling out of complete and overcomplete numbers to the early techniques of manipulation of aliquot parts.[105] The last entry in the so–called 2:*n* table in the Rhind papyrus is 2:101, and this is written as a sum of aliquot parts as $\overline{101}+\overline{202}+\overline{303}+\overline{606}$, which amounts to write 2 as $1+\overline{2}+\overline{3}+\overline{6}$ or 1 as $\overline{2}+\overline{3}+\overline{6}$. Clearly, every division in the 2:*n* table could have been performed this way, but the presence in the sum of the same aliquot part to be doubled makes the decomposition totally ineffective. The reasons for which such a decomposition was included are difficult to figure, even if one can observe that, raising $1=\overline{2}+\overline{3}+\overline{6}$ to common denominator, 6=3+2+1 is obtained. More generally, every decomposition of a complete number into the sum of its proper parts gives rise to a decomposition of a unit into aliquot parts. On this basis and taking for granted Becker's reconstruction outlined above, Caveing advocates the view of an early Greek acquaintance with complete numbers.

3. Assessment

The passage in the *Theaetetus* supports a claim that any historian of ancient mathematics would bet is true: complete numbers were known well before Euclid. Complete numbers are mathematical objects whose early invention was to be expected, Greek arithmetic and logistic showing such an early and strong interest in parts of

[102] Arithmetic and logistic seem to be kept distinct by Plato (at e.g *Gorgias* 451A–C and *Charmides* 166A), and this suggests that both fields had been developed independently and to a reasonable extent, but it must be said that the available sources do not allow to draw any sharp borderline between the two domains. Klein 1968, in particular chapter I.3, following the definitions in Plato, maintains that pre–Euclidean arithmetic was concerned with numbers in themselves, whereas logistic allowed for relations (e.g. operations and ratios) among them. Accordingly, both sciences had a theoretical and a practical side. Admittedly, with Klein's definition of logistic it is not clear what was left to arithmetic to deal with.

[103] Cfr. fragments DK 47 A 16, 17, and 19, as well as DK 47 B 2 and the recently proposed Archytean fragment in Ptolemaeus, *Harmonica*, pp. 11.8–12.7 and 12.24–27 (Düring) (see Barker 1994).

[104] See Caveing 1994, vol. 1, pp. 356–359, and vol. 2, pp. 222–226.

numbers. In particular, the present analysis appears to support either of Taisbak's and Becker's reconstructions, in that the very conception of complete numbers as theoretical objects can hardly be disjoined from detailed investigations about an effective way of generating them.

One could wonder why the invention of complete numbers went anonymous. It could have been easy, most notably to the late Neo–Pythagoreans, to attach a name to them. The fact that they did not do that could simply mean that they did not know of any name to attach to. The very limited number of sources in which Theaetetus' mathematical achievements are attested suggests in fact that he was no longer well–known in late antiquity, perhaps only through Plato's dialogue and the *Histories* of Eudemus. On the other hand, the same silence on the name, despite the relevance the Neo–Pythagoreans granted to the issue of complete numbers, could be interpreted as suggesting that the invention of the concept was notoriously not to be ascribed to a Pythagorean, and Theaetetus was in fact not renowned as such.[106] Late authors had therefore to forge a fictitious ascription. Not surprisingly, Iamblichus claims in fact that friendly numbers (and by implication also complete numbers) were discovered by Pythagoras: he clearly cannot assign the invention of complete numbers to Theaetetus. A similar phenomenon could have originated the partial eclipse of Theaetetus' name as the inventor of the concept of regular solid. Such an invention is ascribed to Theaetetus just in a couple of sources, independent of Neo–Pythagorean tradition, and in them in a very succint and casual way. The Neo–Pythagoreans firmly maintained instead that the discovery of the regular solids and the related cosmology expounded in the *Timaeus* were obviously due to Pythagoras: no room was left in their narratives for Theaetetus.[107]

Moreover, the transmission of the name of a mathematician who opened a field of research is made far easier if the same field is kept alive by centuries of subsequent investigations. This did not happen with ancient arithmetic, that appears to have turned from abstract number theory as we have in the *Elements* both to indeterminate analysis and to a refinement of calculation tools, the latter mainly after the introduction of the

[105] See Hultsch 1895, in particular pp. 156–166, Tannery 1900, p. 33-35, referring to Hultsch, and Itard 1961, pp. 69–70, quoting Tannery who refers to Hultsch.
[106] Theaetetus' master Theodorus is included by Iamblichus in his catalogue of Pythagoreans (*De Vita Pythagorica* 267), but recall that in the *Theaetetus* Theodorus is the spokesman of Protagoras' philosophy. At 145C–D Socrates asserts that Theaetetus is learning geometry, astronomy, harmonics, and logistic from Theodorus. These are the disciplines composing what lately became the Pythagorean *quadrivium*. This could well have sufficed for Theodorus to be reckoned among the Pythagoreans.
[107] A discussion of the testimonies is in Sachs 1917. See also Euclide 2001, pp. 95–106.

sexagesimal system.[108] This on good grounds: *Elements* IX.36 is a mathematical *cul de sac*, and later reports on the issue are muddled by numerological overtones. Thus IX.36 is a wonderful theorem, but appears to be very well suited to close a field of research.

A further reason for ancient and present ignorance about the issue of complete numbers before Euclid is that no ancient commentaries survive to the arithmetic books of the *Elements*. We know that Hero commented on them, since he is the only quoted source in the few pages devoted to the arithmetic books in an–Nayrîzî's commentary.[109] Unfortunately, the first leaf of this section of an–Nayrîzî's text had already got lost when Gerard of Cremona translated the work,[110] and the Arabic text we have in the *Codex Leidensis* 399.1 stops after a few definitions of book VII. As a consequence, no comments on the definitions have been transmitted. The remaining remarks and integrations ascribed to Hero hardly carry any exciting content.[111] The long section devoted to IX.36 amounts to a slightly reworked translation of the proof.[112] The author of the *Definitiones* transmitted in the Heronian *corpus* asserts twice to have composed *Preliminaries to the Arithmetic Elements*.[113] We have nothing of this commentary. We have just one fragment from Eudemus' *History of Arithmetic,* but it deals with a rather trifling application of arithmetic to harmonics.[114] (Recall that much of what we know about pre–Euclidean arithmetic is a by–product of an uninterrupted interest in harmonic theory.[115] For instance, the most important arithmetic fragment of Archytas is preserved in an incidental remark in Boethius.[116]) We must conclude that arithmetic suffered more than geometry from the peculiarities of the process of transmission of ancient mathematical knowledge: the *Elements* eclipsed nearly every preceding production, the late Neo–Pythagorean syntheses, while replacing more reliable sources, gave the

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[108] Late developments of combinatorics should be added: see Acerbi 2003.

[109] Anaritius, *In decem libros priores Elementorum Euclidis commentarii*, pp. 190–210 (Curtze).

[110] Gerard asserts this at the very beginning of the translation of book VII. See Anaritius, *In decem libros priores Elementorum Euclidis commentarii*, p. 190.1–2 (Curtze).

[111] Actually, the way the several comments of an–Nayrîzî on the arithmetic books are reported suggests that all of them come from Hero's commentary.

[112] Curtze found this proof and a proof of IX.13 among the leaves containing the commentary to book X. He suggests that the proofs could be fragments of the original translation by Gerard (see Anaritius, *In decem libros priores Elementorum Euclidis commentarii*, p. 200 note 1 (Curtze)). However, the translations *do not* coincide with the ones in Busard 1984, cc. 218.36–220.6 and 230.44–232.37 respectively.

[113] See Heron, *Definitiones*, Sects. 122 and 128, pp. 76.21–78.2 and 84.17–22 (Heiberg). Cfr. the discussion in Knorr 1993.

[114] Porphyrius, *In Ptolemaei Harmonica commentarium*, p. 115.4–9 (Düring) (= fr. 142 Wehrli).

[115] Outside the *Elements*, the only conspicuous and consistent tract on early Greek number theory is preserved as the first nine propositions of the *Sectio canonis*.

[116] *Institutio musica* III.11, pp. 285–86 (Friedlein).

historically–inclined reports a turn that was not suited to preserve unbiased records. Therefore, even if Theaetetus did invent complete numbers, absence of references to this fact in ancient authors need not surprise us. The lack of direct sources has forced many historians to advocate different forms of the odd historiographical position that the arithmetic books of the *Elements* were simply the passive repository of pre–Euclidean number theory. As we have seen, conspicuous portions of the *Elements* are ascribed lock, stock, and barrel to authors who lived more than one century before Euclid. Maybe a more firm support on contemporary sources would be desirable, otherwise suspension of judgement is the safest attitude.

I feel that the present study is more properly regarded as pointing to a reappraisal of a certain kind of sources than as providing new insights on ancient Greek mathematics. Ascribing a well–known but anonymous result to a well–known mathematician can hardly be considered as an original historiographical move, if not because scholars have spent a certain amount of their scholarship in the last 40 years in sifting sound ascriptions from unsound ones in the garble of Peripatetic and Neo–Pythagorean narratives of past mathematics. My proposal shows that two chunks in an otherwise very disconnected puzzle could match, but gives little contribution to the reconstruction of even a part the whole puzzle. In particular, the hypothesis of an early transfer of Egyptian methods to Greece can be brought to bear on the issue only via the intermediation of Taisbak's reconstruction.

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Sources

Anaritius, *In decem libros priores Elementorum Euclidis commentarii,* edidit M. Curtze. Leipzig, B. G. Teubner 1899 (supplementum ad *Euclidis Opera Omnia,* ediderunt I. L. Heiberg et H. Menge. Leipzig, B. G. Teubner 1883–1916).

Anonymous in Theaetetum, PBerol inv. 9782, G. Bastianini, D. N. Sedley eds., in *Corpus dei papiri filosofici greci e latini*. Parte III: Commentari, pp. 227–562. Firenze, Leo S. Olschki 1995.

Aristotle, *Aristotle's Metaphysics*. A Revised Text with Introduction and Commentary by W. D. Ross. 2 vols. Oxford, Clarendon Press 1924.

Aristotle, *Aristotle's Physics*. A Revised Text with Introduction and Commentary by W. D. Ross. Oxford, Clarendon Press 1936.

(Pseudo–)Aristoteles, *De lineis insecabilibus*. Introduzione, traduzione e commento a cura di M. Timpanaro Cardini. Milano and Varese, Istituto Editoriale Cisalpino 1970.

H. L. L. Busard, *The Latin translation of the Arabic version of Euclid's* Elements *commonly ascribed to Gerard of Cremona.* Leiden, Brill 1984.

Davidis Prolegomena et in Porphyrii Isagogen commentarium, in *Commentaria in Aristotelem Graeca*, vol. XVIII.2, ed. A. Busse. Berlin (Academia litterarum regia borussica), G. Reimer 1904 (reprint ed.: Berlin, De Gruyter 1957).

H. Diels, W. Kranz, *Die Fragmente der Vorsokratiker*. 6th ed. 3 vols. Berlin, Weidmann 1951 (reprint ed.: Zürich, Weidmann 1996).

Diogenes Laertius, *Vitae Philosophorum*, ed. H. S. Long, 2 vols. Oxford, Oxford University Press 1964.

Diophantus Alexandrinus, *Opera Omnia cum Graeciis commentariis*, edidit et latine interpretatus est P. Tannery. 2 vols. Leipzig, B. G. Teubner 1903–1905 (reprint ed.: Stuttgart, B. G. Teubner 1974).

Die Harmonielehre des Klaudios Ptolemaios, herausgegeben von I. Düring. Göteborgs Högskolas Årsskrift XXXVI. Göteborg, Elanders Boktryckeri Aktiebolag 1930 (reprint ed.: Hildesheim, Georg Olms Verlag 1982).

Porphyrios Kommentar zur Harmonielehre des Ptolemaios, herausgegeben von I. Düring. Göteborgs Högskolas Årsskrift XXXVIII. Göteborg, Elanders Boktryckeri Aktiebolag 1932 (reprint ed.: Hildesheim, Georg Olms Verlag 1978).

Euclides, *Elementa*, post I. L. Heiberg edidit E. S. Stamatis. 5 vols. Leipzig, B. G. Teubner 1969–1977.

G. Frege, *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl*. Breslau, W. Köbner 1884 (reprint ed.: Hildesheim, Georg Olms Verlag 1990).

Heron, *Heronis Alexandrini opera quae supersunt omnia. Volumen IV, Heronis Definitiones cum variis collectionibus*, *Heronis quae feruntur Geometrica*, edidit I. L. Heiberg. Leipzig, B. G. Teubner 1912 (reprint ed.: Stuttgart, B. G. Teubner 1976).

Iamblichus, *In Nichomachi arithmeticam introductionem liber*, edidit H. Pistelli. Leipzig, B. G. Teubner 1894 (reprint ed.: Stuttgart, B. G. Teubner 1975).

[Iamblichus], *Theologoumena arithmeticae*, edidit V. De Falco. Leipzig, B. G. Teubner 1922 (reprint ed.: Stuttgart, B. G. Teubner 1975).

Iamblichus, *De Vita Pythagorica*, edidit L. Deubner. Leipzig, B. G. Teubner 1937 (reprint ed.: Stuttgart, B. G. Teubner 1975).

I. Kant, *Kritik der reinen Vernunft*, Riga, Hartknoch 1787[2].

Hippolytus, *Refutatio omnium haeresium*, edidit M. Marcovich. Patristische Texte und Studien 25. Berlin und New York, W. De Gruyter 1986.

Musici scriptores graeci, recognovit C. Jan. Leipzig, B. G. Teubner 1895 (reprint ed.: Stuttgart und Leipzig, B. G. Teubner 1995).

Nicomachus, *Introductio arithmetica*, edidit R. Hoche. Leipzig, B. G. Teubner 1866.

Pappus, *Collectionis quae supersunt*, edidit F. Hultsch. 3 vols. Berlin, Weidmann 1876–78 (reprint ed.: Amsterdam, Hakkert 1965).

Pappus, *Commentary on Book X of Euclid's Elements*. Arabic text and translation by W. Thomson with introductory remarks, notes, and a glossary of technical terms by G. Junge and W. Thomson. Cambridge (Mass.), Harvard University Press 1930.

Plato, *Platonis Opera*, Tomus I, Tetralogias I–II continens, recognoverunt brevique adnotatione critica instruxerunt E. A. Duke, W. F. Hicken, W. S. M. Nicoll, D. B. Robinson et J. C. G. Strachan. Oxford, Oxford University Press, 1995.

Proclus, *In Primum Euclidis Elementorum Librum Commentarii*, ex recognitione G. Friedlein. Leipzig, B. G. Teubner 1873 (reprint ed.: Hildesheim, Georg Olms Verlag 1992).

L. Tarán, *Speusippus of Athens. A Critical Study with a Collection of the Related Texts and Commentary*. Leiden, Brill 1981.

Scholia Platonica, by W. C. Greene. Monograph Series 8, The American Philological Association 1938 (reprint ed.: Chico (CA), Scholars Press 1981).

Suidae Lexicon, edidit A. Adler, in *Lexicographi Graeci*, vol. I. 5 parts. Leipzig, B. G. Teubner 1928-38 (reprint ed.: München und Leipzig, K.G. Saur 1994-2001).

Theon Smyrnaeus, *Expositio rerum ad legendum Platonem utilium*, ed. E. Hiller. Leipzig, B. G. Teubner 1878 (reprint ed.: Stuttgart and Leipzig, B. G. Teubner 1995).

G. L. Westerink, *Pseudo–Elias* (*Pseudo–David*)*: Lectures on Porphyry's Isagoge*. Amsterdam, North Holland 1967.

L. Wittgenstein, *Philosophische Untersuchungen*. Oxford, Basil Blackwell 1953.

Secondary literature

F. Acerbi 2003, On the Shoulders of Hipparchus. A Reappraisal of Ancient Greek Combinatorics, *Archive for History of Exact Sciences* **57**, pp. 465–502.

J. Annas 1976, *Aristotle's* Metaphysics. *Books M and N.* Translated with Introduction and Notes by Julia Hannas. Oxford, Clarendon Press.

A. Barigazzi 1963, Il Dionysos di Euforione, in *Miscellanea di Studi Alessandrini. In Memoria di A. Rostagni*. Torino, Bottega d'Erasmo, pp. 416–54.

A. D. Barker 1994, Ptolemy's Pythagoreans, Archytas, and Plato's conception of mathematics, *Phronesis* **39**, pp. 113–135.

O. Becker 1936, Die Lehre vom Geraden und Ungeraden im Neunten Buch der Euklidischen Elemente, *Quellen und Studien zur Geschichte der Mathematik, Astronomie, und Physik* **B3**, pp. 533–553.

A. C. Bowen and B. R. Goldstein 1988, Meton of Athens and Astronomy in the Late fifth Century B.C., in E. Leichty, M. deJong Ellis, and P. Gerardi (eds.), *A Scientific Humanist: Studies in Memory of Abraham Sachs*. (Occasional Publications of the Samuel Noah Kramer Fund, 9) Philadelphia, pp. 39–81.

W. Burkert 1972, *Lore and Science in Ancient Pythagoreanism*. Cambridge (Mass.), Harvard University Press. (English translation of: *Wisheit und Wissenschaft: Studien zu Pythagoras, Philolaus und Platon*. Nürnberg, Verlag Hans Carl 1962.)

M. F. Burnyeat 1978, The Philosophical Sense of Theaetetus' Mathematics, *Isis* **69**, pp. 489–513.

M. F. Burnyeat 1990, *The Theaetetus of Plato*, with a translation of Plato's *Theaetetus* by M. J. Levett revised by Myles Burnyeat. Indianapolis and Cambridge, Hackett Publishing Company.

M. Caveing, *La constitution du type mathématique de l'idéalité dans la pensée grecque.* Lille, Presses Universitaires du Septentrion. Vol. 1. *Essai sur le savoir mathématique dans la Mésopotamie et l'Egypte anciennes* (1994). Vol. 2. *La figure et le nombre* (1997).

M. Clagett 1999, *Ancient Egyptian Science: A Source Book. Volume Three: Ancient Egyptian Mathematics*. Memoirs of the American Philosophical Society, vol. 232. Philadelphia, American Philosophical Society.

R. D. Dicks 1959, Thales, *Classical Quarterly* **9**, pp. 294–309.

M. Dummett 1959, Wittgenstein's Philosophy of Mathematics, *Philosophical Review* **68**, pp. 324–48 (reprinted e.g. in P. Benacerraf, H. Putnam (eds.), *Philosophy of Mathematics: Selected Readings*. Englewood Cliffs (N.J.), Prentice Hall 1964, pp. 491–509 and in M. Dummett, *Truth and Other Enigmas*. Cambridge (Mass.), Harvard University Press 1978, pp. 166–185).

Euclide, *Les Éléments*. Traduction et commentaires par Bernard Vitrac. Paris, Presses Universitaires de France. Vol. 1. Livres I à IV (1990), Vol. 2. Livres V à IX (1994), Vol. 3, livre X (1998), Vol. 4. Livres XI à XIII (2001).

D. H. Fowler 1992, Logistic and Fractions in Early Greek Mathematics: a New Interpretation, in P. Benoit, K. Chemla, J. Ritter (eds.), *Histoire des fractions, fractions d'histoire*. Basel, Birkhäuser, pp. 133–147.

D. H. Fowler 1999, *The Mathematics of Plato's Academy*. Oxford, Clarendon Press (2nd ed.).

K. Gaiser 1968, *Platons ungeschriebene Lehre*. *Studien zur systematischen und geschichtlichen Begründung der Wissenschaften in der Platonischen Schule*. Stuttgart, Ernst Klett.

V. Harte 2002, *Plato on Parts and Wholes: the Metaphysics of Structure*. Oxford, Oxford University Press.

J. Høyrup 1990a, *Dynamis*, the Babylonians, and Theaetetus 147c7–148d7, *Historia Mathematica* **17**, pp. 201–222.

J. Høyrup 1990b, On Parts of Parts and Ascending Continued Fractions: An Investigation of the Origins and Spread of a peculiar System, *Centaurus* **33**, pp. 293–324.

J. Høyrup 1996, 'The Four Sides and the Area'. Oblique Light on the Prehistory of Algebra, in R. Calinger (ed.), *Vita mathematica. Historical Research and Integration with Teaching*. Washington (D. C.), Mathematical Association of America, pp. 45–65.

J. Høyrup 1997, Hero, Ps.–Hero, and Near Eastern Practical Geometry. An Investigation of *Metrica*, *Geometrica*, and other Treatises, in K. Döring, B. Herzhoff, and G. Wöhrle (eds.), *Antike Naturwissenschaft und ihre Rezeption*. Band 7. Trier, Wissenschaftlicher Verlag Trier, pp. 67–93.

J. Høyrup 2004, Conceptual Divergence – Canons and Taboos – and Critique, *Historia Mathematica* **31**, pp. 129-147.

C. A. Huffman 1993, *Philolaus of Croton: Pythagorean and Presocratic*. Cambridge, Cambridge University Press.

F. Hultsch 1895, *Die Elemente der ägyptischen Teilungsrechnung. Erste Abhandlung*. Abhandlungen der Königlich–Sächsischen Gesellschaft der Wissenschaften, philologisch–historische Klasse **17.1**. Leipzig, pp. 1–192.

J. Itard 1961, *Les livres arithmétiques d'Euclide*. Paris, Hermann.

A. Jones 1993, Evidence for Babylonian Arithmetical Schemes in Greek Astronomy, in H. D. Galter (ed.), *Die Rolle der Astronomie in den Kulturen Mesopotamiens. Beiträge zum 3. Grazer Morgenländische Symposion (23.–27. September 1991)*. Graz, Grazer Morgenländische Studien 3, pp. 77–94.

J. Klein 1968, *Greek Mathematical Thought and the Origin of Algebra*. Cambridge (Mass.), MIT Press. (English translation of: Die griechische Logistik und die Entstehung der Algebra, *Quellen und Studien zur Geschichte der Mathematik, Astronomie, und Physik* **B3** (1934–1936), pp. 18–105; 122–235.)

W. R. Knorr 1975, *The Evolution of the Euclidean Elements*. Dordrecht and Boston, D. Reidel.

W. R. Knorr 1982, Techniques of Fractions in Ancient Egypt and Greece, *Historia Mathematica* **9**, pp. 133–171.

W. R. Knorr 1993, *Arithmêtikê stoicheiôsis*: On Diophantus and Hero of Alexandria, *Historia Mathematica* **20**, pp. 180–192.

J. L. Lightfoot 1998, An Early reference to Perfect Numbers? Some Notes on Euphorion SH 417, *Classical Quarterly* **48**, pp. 187–194.

I. Mueller 1981, *Philosophy of Mathematics and Deductive Structure in Euclid's* Elements. Cambridge (Mass.) and London, MIT Press.

I. Mueller 1991, On the Notion of a Mathematical Starting Point in Plato, Aristotle, and Euclid, in A. C. Bowen (ed.), *Science and Philosophy in Classical Greece*. New York and London, Garland Publishing, pp. 59–97.

O. Neugebauer 1934–36, Zur geometrischen Algebra (Studien zur Geschichte der antiken Algebra III), *Quellen und Studien zur Geschichte der Mathematik, Astronomie, und Physik* **B3**, pp. 245–259.

J. Ritter 2002, Closing the eye of Horus: The Rise and Fall of ‘Horus–eye Fractions’, in J. M. Stelle, A. Imhausen (eds.), *Under One Sky. Astronomy and Mathematics in the Ancient Near East*. Münster, Ugarit–Verlag, pp. 297–323.

E. Sachs 1917, *Die fünf platonischen Körper. Zur Geschichte der Mathematik und der Elementenlehre Platons und der Pythagoreer*. Berlin, Weidmann.

C. M. Taisbak 1976, Perfect Numbers: A Mathematical Pun? *Centaurus* **20**, pp. 269–275.

C. M. Taisbak 2002, Euclid's Elements 9.14 and the Fundamental Theorem of Arithmetic, in C. J. Tuplin, T. E. Rihll (eds.), *Science and Mathematics in Ancient Greek Culture*. Oxford, Oxford University Press.

P. Tannery 1887, *La géométrie grecque*. Paris, Gauthier–Villars (reprint ed.: Paris, Editions Jacques Gabay 1988).

P. Tannery 1900, Histoire des Sciences, Mathématiques, *Revue de Synthèse historique* **1**, pp. 179-195, reprinted in Id., *Mémoires Scientifiques*, publiés par J.L. Heiberg & H.G. Zeuthen. 17 tomes. Toulouse, Privat/Paris, Gauthier–Villars 1912–1950, tome X (1930), édité avec la collaboration de J. Pérès, n. 3, pp. 15–36.

P. Tannery 1904, A propos des fragments philolaïques sur la musique, *Revue de philologie* **28**, pp. 233–49, reprinted in Id., *Mémoires Scientifiques*, publiés par J.L. Heiberg & H.G. Zeuthen. 17 tomes. Toulouse, Privat/Paris, Gauthier–Villars 1912–1950 (reprint ed. Paris, Editions Jacques Gabay 1995), tome III (1915), n. 82, pp. 220–43.

B. Vitrac, Les formules mathématiques de la puissance (δύναμις, δύνασθαι), forthcoming.

B. Vitrac 1992, Logistique et fractions dans le monde hellénistique, in P. Benoit, K. Chemla, J. Ritter (eds.), *Histoire de fractions, fractions d'histoire*. Basel, Birkhäuser, pp. 149–172.

B. L. van der Waerden 1954, *Science Awakening*. Groningen, Noordhoff.

W. C. Waterhouse 1972–73, The Discovery of the Regular Solids, *Archive for History of Exact Sciences* **9**, pp. 212–221.

G. L. Westerink 1960, A new fragment of Euphorion, *Mnemosyne* **13**, pp. 329–30.

H. Zeuthen 1910, Sur la constitution des livres arithmétiques des Eléments d'Euclide et leur rapport à la question de l'irrationalité, *Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger*, pp. 395–435.

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