IF-logic and the Foundations of Mathematics

Gabriel Sandu

1 Some anomalies: IF logic and the foundations of mathematics

1.1 Hintikka's claims

In PMR Hintikka claims that, despite its semantical incompleteness (property (1) in section 1), IF logic is a better tool for formulating descriptively complete nonlogical theories. (p. 97).

The distinction between different variants of completeness, which originated in the works of Hilbert, is worth exploring in some details.

(a) *Descriptive completeness*: It is the property of a theory which is true exactly in the intended models. When there is only one such model (up to isomporphism), then T is said to be categorical. No proof system is involved in this case.

(b) *Semantic completeness*: It is the property of a proof system $\vdash$ such that for any theory T, all logical consequences of T can be obtained as theorems from T ($T \models \varphi \Rightarrow T \vdash \varphi$).

(c) *Deductive (formal) completeness*: It is the property of a theory T based on an underlying proof system. T is deductivelly complete if for every sentence φ, either φ is derivable or $\neg\varphi$ is derivable from T.

In both (b) and (c) it is usually required that the notion of derivation (proof) be effective.

In ordinary first-order logic, there is a natural connection between these three concepts. Assume we have a theory T and that the underlying logic is semantically complete in the sense of (b). Then it is straightforward to show that T is descriptively complete (categorical) if and only if T is deductively complete.

Since the condition of this fact is not fulfilled the equivalence between descriptive and semantic incompleteness is lost.

We gave earlier an example of an IF-theory consisting of a single sentence φ_{inf} which is true exactly in the infinite models.

The notion of a descriptively complete theory has been introduced by Hilbert in ???: it is the property of a theory which is true exactly in its intended models.

Another property very much praised by Hintikka is the definability of truth. It is better to truncate it, for future reference, into several related concepts:

(a) The predicates 'φ is true-in-N' and 'φ is false-in-N' are definable in the IF-language of PA, for any ordinary *first-order* sentence φ in the signature of PA. In other words, there exist IF-formulas $\Phi(x)$ and $\Psi(x)$ in the signature of PA such that

$$N \models^{+} \Phi(\ulcorner\varphi\urcorner \Leftrightarrow N \models^{+} \varphi$$

and

$$N \models^{+} \Psi(\ulcorner\varphi\urcorner \Leftrightarrow N \models^{-} \varphi$$

for every first-order sentence φ.

(b) The predicates 'φ is true-in-N' and 'φ is false-in-N' are definable in the IF-language of PA, for any ordinary *IF*- sentence φ in the signature of PA. In other words, there exist IF-formulas $\Phi^{*}(x)$ and $\Psi^{*}(x)$ in the signature of PA such that

$$N \models^{+} \Phi^{*}(\ulcorner\varphi\urcorner \Leftrightarrow N \models^{+} \varphi$$

and

$$N \models^{+} \Psi^{*}(\ulcorner\varphi\urcorner \Leftrightarrow N \models^{-} \varphi$$

for every IF-sentence φ. We have to remember, however, that

$$N \models^{+} \varphi \Leftrightarrow N \models_{Tarski} \varphi$$

$$N \models^{-} \varphi \Leftrightarrow N \not\models_{Tarski} \varphi$$

for every first-order sentence φ, and even

$$N \models^{+} \varphi \Leftrightarrow N \models_{Tarski} \varphi$$

for every IF-sentence φ but

$$N \models^{-} \varphi \Leftrightarrow N \not\models_{Tarski} \varphi$$

does not hold for every IF-sentence φ.

The fact that IF logic defines its truth-predicate (in certain models) is according to Hintikka a significant result, because such a predicate

> ... is nothing more and nothing less than the declaration of independence of model theory. It shows that one can develop a model theory for the powerful IF first-order languages on the first-order level, *ergo* independently of all questions of sets and set existence. All the quantifiers in the IF first-order version of my truth predicate range over individuals. ...Consequently, all apprehensions concerning the purely logical status of model theory are groundless. Model theory of first-order logic is part of logic, and not a proper part of mathematics. The problems which are caused by the apparent dependence of the model theory of first-order logic on set theory (or on higher-order logic) can thus be solved, and Tarski's course be exorcised. (PMR, p. 129)

The truth-predicate turns out to be useful in formulating axiomatized theories of different mathematical concepts:

> Indeed, as soon as we have a viable notion of truth-in-a-model at our disposal, and also some characterization of the totality of all models of a given language, then we can set up an axiom set for elementary arithmetic hoping realistically that it might then turn out to be incomplete. ...(PMR, p. 97)

As a result, we would be able to use more deductive methods in logic:

> Hence the main post-Gödelian, not to say postmodern, foundational problem is to look for new deductive methods and to analyze them. ...Hence the task I am talking about here is not entirely unlike the task of finding stronger and stronger axioms of set theory. (PMR, p. 99)

We think that the use of the truth predicate in finding stronger and stronger axioms for some of our mathematical concepts which would allow us to use more deductive methods in the foundations of mathematics raises some interesting philosophical problems. We find some of these problems, however, rather puzzling, and therefore think that the use of more deductive methods in case the underlying logic defines truth and falsity in N for *ordinary first-ordinary sentences*, and in addition has the *Separation*, *Compactness* and *Interpolation* Theorems, cannot work.

We will focus our attention on the case in which there is an axiomatic system for natural numbers. Let us assume that this is so, and suppose we have a proof-system $\vdash_{IF}$ in IF-languages for elementary arithmetic.

The idea here is that this proof system is intended to replace set-theory as a foundational theory for elementary arithmetic. Even in this case, it is reasonable to ask what is the relation of this proof-system to set theory (ZFC).

One such relation is soundness, that is, it is reasonable to assume that $\vdash_{IF}$ is sound, i.e., it implies logical consequence. This means

(*) .

We are aware of the fact that one may argue that soundness does not imply (*). Yet we shall assume that soundness implies (*) at least in the case when $\varphi = \bot$, where $\bot$ is a fixed contradictory first-order sentence. Notice that no conservative extension of the first-order proof system can break this assumption, as witnessed by the following proposition.

Proposition 1 *Assume $\vdash_{IF}$ satisfies the following:*

(i) If Σ is a set of first-order sentences and φ is first-order, then $\Sigma \vdash_{IF} \varphi$ implies that φ is provable from Σ ($\Sigma \vdash \varphi$) in the usual first-order system $\vdash$. (In other words, the $IF-$theory is conservative over the ordinary first-order theory).

(ii) If $\varphi(x, y, z, w)$ is first-order, and F and G are function symbols, then

$$\forall x \forall z \varphi(x, F(x), z, G(z)) \vdash_{IF} \forall x \forall z (\exists y/\{z\})(\exists w/\{x, y\})\varphi(x, y, z, w)$$

Notice that this requirement is very intuitive: given the interpretation of $\forall x \forall z(\exists y/\{z\})(\exists w/\{x, y\})\varphi(x, y, z, w)$ as $\exists F \exists G \forall x \forall z \varphi(x, F(x), z, G(z))$,(ii) amounts to a principle of existential generalization for second-order functions.

Let $\varphi_i(x, y, z, w), i < n$ be first-order, and Σ be the set of IF-sentences:
$\Sigma = \{\forall x \forall z(\exists y/\{z\})(\exists w/\{x, y\})\varphi_0, ..., \forall x \forall z(\exists y/\{z\})(\exists w/\{x, y\})\varphi_{n-1}\}$
Then

(*) If $\Sigma \vdash_{IF} \bot$, then one can prove in ZFC that $\Sigma \models \bot$ (i.e. Σ has no model).

Proof: Assume $\Sigma \vdash_{IF} \bot$. Then by (ii)
$\{\forall x \forall z \varphi_0(x, F_0(x), z, G_0(z)), ..., \forall x \forall z \varphi_{n-1}(x, F_{n-1}(x), z, G_{n-1}(z))\} \vdash_{IF} \bot$
,

where F_i, G_i are new function symbols. By (i) the contradiction is provable in the usual first-order proof system. Since these proofs are finite, we can show this in ZFC. Hence by the soundness of the first-order proof system, we can prove in ZFC that
$\{\forall x \forall z \varphi_0(x, F_0(x), z, G_0(z)), ..., \forall x \forall z \varphi_{n-1}(x, F_{n-1}(x), z, G_{n-1}(z))\}$
does not have a model. This means that in ZFC we can also prove that

$\{\forall x\forall z(\exists y/\{z\})(\exists w/\{x,y\})\varphi_0(x,y,z,w),...,\forall x\forall z(\exists y/\{z\})(\exists w/\{x,y\})\varphi_{n-1}(x,y,z,w)\}$ *does not have a model.*

Assuming that the proof system $\vdash_{IF}$ is sound, we now go on and show that such a proof system is not as strong as we would like it to be.

We would like to be able in the new system to talk, e.g. about first-order definable sets of natural numbers.

One of the basic facts in the theory of arithmetical sets is that the class of such sets is closed under complements. In set theory this is expressed by the fact that for all first-order formulas $\varphi(x)$ and $n \in \mathbb{N}$

$$\mathbb{N} \not\models \varphi(n) \Leftrightarrow \mathbb{N} \models \neg\varphi(n) \qquad (*)$$

It is reasonable to ask how to prove such facts in an axiomatization of IF logic using the truth-defining formula, since such an axiomatization is all we would have to work with, after giving up ZFC. Let the formula which defines truth for first-order sentences in $\mathbb{N}$ be Φ, and the formula which defines falsity for first-order sentences in $\mathbb{N}$ be Ψ. In other words we have:

$$\mathbb{N} \models \Phi(\ulcorner\psi\urcorner) \Leftrightarrow \mathbb{N} \models \psi$$

and

$$\mathbb{N} \models \Psi(\ulcorner\psi\urcorner) \Leftrightarrow \mathbb{N} \not\models \psi$$

for all first-order sentences ψ. From our discussion in section 2, it is obvious that such formulas exist.

Let $\perp$ be as above, $Sen(x)$ the IF-formula stating that x codes a first-order sentence and c a new constant. Let also T be the IF theory axiomatizing the theory of $\mathbb{N}$ (actually T can be chosen to be the set of all IF sentences true in $\mathbb{N}$).

Now assuming something like (*) is provable, the theory

$$T \cup \{Sen(c), \Phi(c), \Psi(c)\}$$

should be contradictory under the assumptions made above. Thus we expect that

$$T \cup \{Sen(c), \Phi(c), \Psi(c)\} \vdash_{IF} \perp .$$

However, this cannot be so, as the following fact shows:

Proposition 2 $T \cup \{Sen(c), \Phi(c), \Psi(c)\} \not\vdash_{IF} \perp$

Proof: Because $\vdash$ *is assumed to be sound, it is enough to find a model* $\mathcal{A}$ *such that*

$$\mathcal{A} \models T \cup \{Sen(c), \Phi(c), \Psi(c)\}$$

For a contradiction, assume that there is no such model. By the compactness of IF-logic, we can find a finite conjunction Θ of sentences from T such that

$$\{\Theta \wedge Sen(c) \wedge \Phi(c), \Psi(c)\}$$

is inconsistent. By the Interpolation theorem for IF logic, we can find a first-order sentence $\varphi(c)$ such that

$$\Theta \wedge Sen(c) \wedge \Phi(c) \models \varphi(c) \text{ and } \varphi(c) \models \neg\Psi(c) \qquad (**)$$

But now it can be shown that $\varphi(x)$ defines $truth - in - \mathbb{N}$ for first order sentences, i.e.

$$\mathbb{N} \models \varphi(\ulcorner\psi\urcorner) \Leftrightarrow \mathbb{N} \models \psi$$

for every first-order sentence ψ. In order to see this, assume first $\mathbb{N} \models \varphi(\ulcorner\psi\urcorner)$. Hence, from the right conjunct of (**), we get $\mathbb{N} \models \neg\Psi(\ulcorner\psi\urcorner)$, and since $\Psi(x)$ is the falsity defining formula, we infer $\mathbb{N} \models \psi$. For the converse, assume $\mathbb{N} \models \psi$, hence $\mathbb{N} \models \Theta \wedge Sen(c) \wedge \Phi(c)$, and thus from (*) we get $\mathbb{N} \models \varphi(\ulcorner\psi\urcorner)$.$\square$

We can actually formulate this result by relating it explicitly to the difficulty of finding a complement:

Proposition 3 *Assume $\psi(x, y)$ is a first-order formula and in the theory T we can prove the following: 'ψ defines a function f_ψ and $Sen(x)$ implies $Sen(f_\psi(x))$'. If for all first-order sentences θ*

$$\mathbb{N} \not\models \theta \Leftrightarrow \mathbb{N} \models \Phi(f_\psi(\ulcorner\theta\urcorner))$$

then $T \cup \{Sen(c), \Phi(c), \Phi(f_\psi(c))\} \not\vdash_{IF} \bot$.

Proof: Immediate from Proposition 5. $\square$

Proposition 4 *(Bozon) There is no sound proof system $\vdash_{IF}$ for IF-logic, which is semantically complete, that is:*

$$\Sigma \models \varphi \Rightarrow \Sigma \vdash_{IF} \varphi$$

for every set Σ of IF-sentences, and IF-sentence φ, such that $\vdash_{IF}$ is an effective proof system.

Proof: Recall our set of sentences $\{\varphi_n : n \geq 2\}$ and the sentence φ_{inf}. We saw that

$$\{\varphi_n : n \geq 2\} \models \varphi_{\text{inf}}$$

On the other side, we cannot have $\{\varphi_n : n \geq 2\} \vdash_{IF} \varphi_{\text{inf}}$, because by the effectiveness of $\vdash_{IF}$ there would be a finite $\Gamma \subset \{\varphi_n : n \geq 2\}$ such that $\Gamma \vdash_{IF} \varphi_{\text{inf}}$, which by the soundness of $\vdash_{IF}$ would imply $\Gamma \models \varphi_{\text{inf}}$, which is impossible. $\square$